DOCUMENT RESUME

ED 137 115 SE 022 341

AUTHOR Lindvall, C. M.; And Others
TITLE The LRDC Individualized Mathematics Project: Program Content, Management System, and Procedures in Development and Implementation.
INSTITUTION Pittsburgh Univ., Pa. Learning Research and Development Center.
SPONS AGENCY National Inst. of Education (DHEW), Washington, D.C.
REPORT NO LRDC-1976-12
PUB DATE 76
NOTE 99p.; For related documents, see SE 022 339-340; Not available in hard copy due to small print throughout entire document

EDRS PRICE MF-$0.83 Plus Postage. HC Not Available from EDRS.
DESCRIPTORS Activity Learning; *Curriculum Development; Elementary Education; *Elementary School Mathematics; Evaluation; *Individualized Instruction; Instruction; Mathematics Curriculum; Mathematics Education; *Program Descriptions
IDENTIFIERS Learning Research and Development Center

ABSTRACT

The development of the Individualized Mathematics (IM) program is described in this paper. This program, for use in grades K-3, incorporates elements from predecessor Learning Research and Development Center mathematics programs, IPI mathematics, and PEP quantification. It features the systematic use of lessons involving manipulatives in an individualized program and the gradual introduction of paper-and-pencil lessons. The paper describes the overall structure of the program, including lesson and test materials and classroom management procedures. It provides a rationale for the mathematics taught in the program by listing the pertinent descriptions and definitions associated with selected mathematical concepts, together with examples of instructional objectives derived from such definitions. Selected evaluation data are provided, and some conclusions concerning the general tasks of development and implementation are discussed. (Author/DT)

****
* Documents acquired by ERIC include many informal unpublished *
* materials not available from other sources. ERIC makes every effort *
* to obtain the best copy available. Nevertheless, items of marginal *
* reproducibility are often encountered and this affects the quality *
* of the microfiche and hardcopy reproductions ERIC makes available *
* via the ERIC Document Reproduction Service (EDRS). EDRS is not *
* responsible for the quality of the original document. Reproductions *
* supplied by EDRS are the best that can be made from the original. *
****

LEARNING RESEARCH AND DEVELOPMENT CENTER

SE

1976/12

THE LRDC MATHEMATICS PROJECT: PROGRAM CONTENT, MANAGEMENT SYSTEM, AND PROCEDURES IN DEVELOPMENT AND IMPLEMENTATION

C.M. LINDVALL, GWENDOLYN DILLARD, ALICE HOSTICKA, JOSEPH LA PRESTA, JUDY LIGHT, M. KATHRYN MEESE, GEORGE MILLER, AND KATHRYN PIKULA

U.S. DEPARTMENT OF HEALTH, EDUCATION & WELFARE
NATIONAL INSTITUTE OF EDUCATION

THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL NATIONAL INSTITUTE OF EDUCATION POSITION OR POLICY

ED137115

SE 022 341

University of Pittsburgh

THE LRDC INDIVIDUALIZED MATHEMATICS PROJECT: PROGRAM CONTENT, MANAGEMENT SYSTEM, AND PROCEDURES IN DEVELOPMENT AND IMPLEMENTATION

C. M. Lindvall
Gwendolyn Dillard
Alice Hosticka
Joseph LaPresta
Judy Light
M. Kathryn Meese
George Miller
Kathryn Pikula

Learning Research and Development Center

University of Pittsburgh

1976

The research reported herein was supported by the Learning Research and Development Center, supported in part as a research and development center by funds from the National Institute of Education (NIE), United States Department of Health, Education, and Welfare. The opinions expressed do not necessarily reflect the position or policy of NIE, and no official endorsement should be inferred.

Abstract

This paper describes the work of LRDC staff members in developing the Individualized Mathematics (IM) program. This program, for use in grades K-3, incorporates elements from predecessor LRDC math programs, IPI mathematics, and PEP quantification. It features the systematic use of manipulative lessons in an individualized program and the gradual introduction of paper-and-pencil lessons. The paper describes the overall structure of the program, including lesson and test materials and classroom management procedures. It provides a rationale for the mathematics taught in the program by listing the pertinent descriptions and definitions associated with selected mathematical concepts, together with examples of instructional objectives derived from such definitions. The paper also provides certain types of evaluation data and some conclusions concerning the general tasks of development and implementation as derived from the three-year experience of the project staff.

TABLE OF CONTENTS

INTRODUCTION 1
Goals of the IM Math Project 3

THE MATHEMATICAL CONTENT OF THE IM PROGRAM 6
The IM Program and "Modern Mathematics" 8
The Structure of Mathematical Systems 10
Summary 27

THE PLAN AND STRUCTURE OF THE MANAGEMENT SYSTEM OF THE IM PROGRAM 28
The Structure and Sequence of Objectives and Units 28
The Testing Program 31
Lesson Materials 34
Prescription Development Procedures 38
Classroom Management Procedures 40
Pupil Classroom Activities 45
Summary 46

THE ROLE OF EVALUATION IN THE DEVELOPMENT OF THE IM PROGRAM 47
Evaluating the Worth of Program Goals 48
Evaluating the Program Plan 48
Evaluating the Program in Operation 48
Assessing the Achievement of Program Goals 53
Some Descriptive Evaluation Data 55

THE PROCESS OF DEVELOPMENT AND IMPLEMENTATION: SOME CONCLUSIONS FROM WORK ON THE IM PROJECT 61
The Tryout and Modification of Materials as an Aspect of In-School Development 63
System Implementation and Mofication as Aspects of In-School Development 65
Summary 70

REFERENCE NOTES 71

REFERENCES 72

APPENDIX A: Specific Objectives for the Individualized Mathematics (IM) Program as Organized by Units 74

THE LRDC INDIVIDUALIZED MATHEMATICS PROJECT: PROGRAM CONTENT, MANAGEMENT SYSTEM, AND PROCEDURES IN DEVELOPMENT AND IMPLEMENTATION

C. M. Lindvall, Gwendolyn Dillard, Alice Hosticka, Joseph LaPresta, Judy Light, M. Kathryn Meese, George Miller, and Kathryn Pikula

Learning Research and Development Center
University of Pittsburgh

Work at the Learning Research and Development Center (LRDC) on the individualization of instruction in mathematics has involved three significant programs: Individually Prescribed Instruction (IPI; Lindvall & Bolvin, 1967), the Primary Education Project (PEP) Quantification Program (Resnick, Wang, & Kaplan, 1973), and Individualized Mathematics (IM). Each of these programs was initiated partially in response to practical needs and represented an effort to investigate certain new ideas as to how an instructional system could be made more effective. Although each had its unique characteristics, all three programs were closely related and have been mutually supportive. Each of the latter two programs built, to some extent, on the programs that preceded it and attempted to correct certain weaknesses in predecessor programs. In a sense, these three programs can be considered as representing three successive stages in the Center's efforts in designing individualized instructional systems.

IPI represented the LRDC's pioneering effort in designing a system for individualizing instruction. The emphasis in its development was on the identification and use of specific system components (placement tests, pretests, curriculum-embedded tests, lesson booklets, prescription

writing procedures, etc.) that permitted a classroom teacher to implement individualized instruction. The general rationale for designing this type of instructional system has been described by Glaser (1965), and the basic principles which guided the identification of specific program components have been discussed by Lindvall and Bolvin (1967).

The PEP program was initiated, partially, as an effort to provide an effective program for use in preschool, kindergarten, and grade one (levels at which the IPI curriculum was either nonexistent or obviously inadequate). More importantly, however, from the point of view of increasing sophistication in LRDC's approach to program design, it introduced significant new components. The design of the PEP curriculum involved the systematic application of behavior analysis procedures and the development of hierarchies of objectives within each unit. Procedures followed in this have been described by Resnick, Wang, and Kaplan (1973). PEP also introduced the planned use of manipulative materials in both lesson materials and tests. What this involves is described in a later section of this report.

The IM program can be considered as an effort to take the most successful components of both IPI and PEP, develop more detailed procedures for classroom management and instruction, and design a program for the primary grades. In part, it developed from a need for extending the use of procedures and materials included in the PEP program into higher levels of instruction (grades 2 and 3) and for providing for a smooth transition from a PEP-type curriculum, with its heavy reliance on manipulative lessons, to a workbook-oriented curriculum, as in the upper levels of the IPI program. Major innovations introduced by the IM project included: (a) the careful integration of manipulative materials with paper-and-pencil exercises to teach a given objective; (b) the production and use of uniform manipulative materials ("boxes") in all classrooms in the primary grades; (c) the development of manuals (e.g., the Teacher's Manual [Lindvall, Note 1] and the Testing Manual

[Lindvall, Note 2]) and guidelines (e.g., "Classroom Management of the Individualized Mathematics Program" [Lindvall, Note 3]) which provide detailed descriptions of the management components of the instructional system; and (d) the attempt to relate the specific instructional objectives of the curriculum to concepts, definitions, and operations representing basic components of the content of modern mathematics. The rationale for the IM project's concern for attention to all components of the instructional program or "system" is contained in "The Learning Research and Development Center Individualized Mathematics Project" (Hosticka, Light, Lindvall, Meese, Miller, & Oles, Note 4). A rather complete analysis of the basic modern math content that underlies the IM program is provided by Hosticka (1973).

Goals of the IM Project

Work on the IM project was directed toward the achievement of certain specified goals. These goals may be categorized according to three types: (a) desired changes in pupils, (b) characteristics of the proposed program, and (c) basic questions to be investigated.

Desired Changes in Pupils

The ultimate, or "payoff," goals for any instructional system consist of the desired changes in pupils that the program is designed to produce. For the IM project, these goals were identified as involving a continuation and extension of the basic goal of IPI and of other LRDC projects, namely, that the program would be "adaptive to the requirements of the individual learner . . . [in such a way as to develop within] the student the favorable attitudes and the abilities which will enhance the probability that he will continue to be a learner throughout his life" (Lindvall & Cox, 1970, p. 34). The IM project identified three general aspects of this basic goal as defining the project goals:

1. The pupil will acquire, in addition to competency in performing arithmetic operations, mastery of mathematics content which emphasizes basic structure, properties, and relations, and will be able to display this mastery at a level of understanding which provides a sound basis for "moving on" in mathematics.

2. The pupil will acquire "study skill" abilities that permit him or her to learn from a variety of types of materials and experiences and will acquire abilities in self-direction and self-evaluation which will enable him or her to be an effective independent learner.

3. The pupil will acquire a "positive attitude" toward learning and toward the study of math.

Process Goals: Characteristics of the Proposed Program

Although payoff goals define the ultimate purpose of a development effort, they seldom provide real direction for development activities. To serve this latter purpose, goals should be spelled out in answer to the question, "What will be the major characteristics of this proposed program if it is to produce the results described by the payoff goals?" These process goals are related to and derived from the payoff goals, but they should describe the essential qualities that will characterize the program when it is completed. They describe the program in terms of the major elements that will determine the "process" to which students are exposed as they study under this system. As such, they provide direction for development efforts and major criteria for assessment of the completed program. The process goals for the IM program include the following:

1. The curriculum will be based on mathematics content which emphasizes an understanding of mathematical systems as well as competency in performing operations.

2. The specific objectives of the curriculum will be organized into hierarchies within units, and units will be organized in a structure of hierarchical relationships, in such a way as to permit identification of prerequisite relationships and alternative instructional paths.

3. The program will provide for some variety in instructional materials, including paper-and-pencil and manipulative activities, and provide for the progressive development of independent study skills.

4. The program will be an instructional system made up of specified components involving materials and specific procedures for pupil and teacher use of the materials.

It should be pointed out that these four process goals represent the relatively unique emphasis of the IM program development effort. Since IM includes many of the components found in PEP and IPI, it shares most of the process goals of these programs.[1] However, these latter goals, dealing largely with the desired characteristics of a system of individualized instruction, did not provide direction for development efforts unique to the IM program and, hence, are not included here.

---

[1] For example, the goals of IPI were stated as: (1) Every pupil makes regular progress towards mastery of instructional content. (2) Every pupil proceeds to mastery of instructional content at an optimal rate. (3) Every pupil is engaged in the learning process through active involvement. (4) The pupil is involved in learning activities that are wholly or partially self-directed and self-selected. (5) The pupil plays a major role in evaluating the quality, extent, and rapidity of his progress towards mastery of successive areas of the learning continuum. (6) Different pupils work with different learning materials and techniques of instruction adapted to individual needs and learning styles. (Lindvall & Cox, 1970, pp. 30-34)

Basic Questions Investigated

Since work on the IM project involved an extension and further trial of program components suggested by earlier LRDC development activities, the staff also thought of its work as involving an examination of questions or hypotheses concerning the usefulness of these components in a program of the scope of IM. These questions may be posed as follows:

1. Can manipulative materials be used by primary grade pupils on an independent study basis under classroom procedures that can be managed by the classroom teacher?
2. Can a hierarchical structure of units of instruction be developed for a relatively large segment of a curriculum (approximately four grade levels, i.e., K-3), and can such a structure be used in a meaningful way by teachers and pupils?
3. What are some of the problems encountered with respect to production, maintenance, storage, and distribution in the broad-scale use of manipulative lesson materials?
4. What are the essential system components with respect to pupil procedures, teacher activities, and general classroom management techniques for this type of individualized instruction?

Answers to these questions are provided in the later sections of this report that describe the management system and the results from the use of evaluation in program development.

THE MATHEMATICAL CONTENT OF THE IM PROGRAM

In developing a curriculum in a subject matter area, the developer must be concerned with the identification of the content that is to be "covered" and with the behavior competencies that a student is to be

expected to acquire with respect to that content. Glaser (1965) has called these concerns the tasks of developing the content repertoire and the component repertoire, respectively. Quite obviously, in the development of a curriculum in an established content area such as primary grade mathematics, the developer is not faced with the task of defining totally new content or component repertoires. Both the existence of established curricula that have shown a degree of success over some period of time and the practical requirement that any new curriculum program be integrated into a total school program that anticipates certain outcomes from a basic skill subject such as mathematics dictate that the developers of a new program give careful attention to what is taught in existing programs. Any new mathematics curriculum then must, in a sense, be a modification and adaptation of existing curricula.

In identifying the units of study and the specific instructional objectives for the IM program, the staff made use of a variety of resources. Major sources of ideas were the IPI Mathematics and PEP Quantification curricula, together with evaluative information concerning the effectiveness of the structures of objectives in these programs. Ideas obtained from these sources were supplemented by a careful reexamination of a variety of other curricula, including the School Mathematics Study Group (SMSG) series and various textbook series somewhat derived from the SMSG experience. The objectives obtained from all of these sources were, in turn, reviewed and edited by relating them to basic definitions of mathematical concepts and operations as provided by mathematicians. This total process was a reflection of the project's concern with the development of a program that facilitated children's learning of mathematics. Building on other curricula and on research and evaluative information that provided some guidelines as to when pupils could learn various concepts and as to the order in which some of these should be taught reflected the concern for effectiveness in children's learning. Editing these objectives on the basis of mathematicians' definitions of

concepts and operations reflected the project's interest in insuring that what was learned was sound mathematics.

In examining this section on the mathematical content of the IM program, then, the reader should not expect to find every concept and definition listed here included in a specific objective of the program. No attempt was made to derive objectives from content definitions through this type of one-to-one relationship. Rather, the content listed here includes those definitions which the staff found to be of key importance in editing their objectives for correctness, completeness, and possible sequential dependency. The content summary was also important as a source for suggesting certain objectives to be added to the curriculum. In addition, it was useful in providing ideas as to how certain topics might be organized and taught. For example, if the definition of a concept indicated that a pupil would ultimately have to understand the concept in a given way, then this provided some guidance as to how the concept should be approached at the primary grade level where less than complete understanding was the goal. Finally, this general description of the content of the IM program gives the interested reader an indication of the mathematics towards which primary grade instruction should be directed.

The IM Program and "Modern Mathematics"

In a general sense, the content of the Individualized Math Program is that which has come to be characterized as "modern mathematics." What is covered in this curriculum has much in common with what is found in the elementary school textbooks prepared by the School Mathematics Study Group (1965) and in most of the more widely used commercial textbooks that propose to teach modern mathematics. Further direction for the LRDC effort has been provided by the rather broad suggestions found in the Report of the Cambridge Conference, Goals for School Mathematics (1963). In general, the overall goal in terms

of pupil content acquisition is to have pupils acquire those abilities and understandings defined by a "modern mathematics" program which will permit the pupil to make an easy transition to a modern junior high and high school program.

The decision to base the content of the new LRDC math program on that generally described as "modern math" was based on considerations such as the following:

1. There appears to be substantial consensus among most of the more widely used textbook series concerning what type of content should be offered in today's elementary school. Much of this consensus appears to derive from an acceptance of the SMSG series as providing overall guidance as to what should be taught.

2. The focus of the LRDC development effort has been on adapting instruction to individual differences in pupils. Because of the particular focus of the IM project, it has looked to other prestigious groups, individuals, and agencies for guidance in identifying the content to be taught. A major resource of this type has to be the work of the SMSG and many related "spin-off" efforts. The hundreds of mathematicians and teachers involved in this effort, the amount of money and number of man-hours invested, and the widespread use of the textbooks and other materials produced made the SMSG math content a basic reference point for any new math program. In the LRDC program, this content can be considered as providing a basic starting point from which Goals for School Mathematics (1963) and many more recent efforts such as those of Dienes (Dienes & Golding, 1971) and others suggest directions for additional content and experiences.

3. "Modern math" is not merely a new way of organizing and teaching elementary and secondary school mathematics. It reflects changes in, and additions to, the total body of knowledge that is known

as mathematics. Wilder (1970), in discussing the characteristic features of modern mathematics, said:

> During the past half-century, new types of structures have been created which not only have furnished means for the solution of long-standing, unsolved problems, but also have provided an instrument without which the student would hardly be able to comprehend the vast accumulation of new mathematics which is being discovered (or invented). (p. 19)

Modern mathematics in the elementary school typically provides for an introduction to such structures as well as to many other aspects of the new body of mathematical knowledge. Not to teach modern math in the elementary school would mean being out of phase with mathematics as a content discipline.

4. Modern mathematics provides a sound and an interesting basis for effective instruction. Because of its emphasis on the structure and properties of such things as the number systems and the operations defined on these systems, the newer approach to math provides a basis for a real understanding of content and procedures. Also, because of its many suggestions for physical representations and manipulations of basic concepts and its emphasis on exploratory and discovery activities, interesting types of instructional procedures can be designed.

5. Identifying a program as being based on "modern mathematics" does not minimize the importance of computational skill nor the importance of being able to use mathematics to solve practical everyday problems. The IM program places significant emphasis on both of these abilities.

The Structure of Mathematical Systems

A substantial portion of the content of the new IM program can be considered as directed toward the pupils achieving comprehension of "the structure of the rational number system." The earlier quotation from Wilder (1970) suggested that the creation of mathematical structures

represents a major contribution to the integration of much of mathematics. In the same article, he goes on to state that teachers have found that such structures "afforded a valuable educational device, while at the same time accomplishing the objective of simplification and consolidation" (p. 21). Near the conclusion of his article, he suggests that "more emphasis will have to be placed on mathematics as the study of structure" (p. 22). As mentioned above, one structure that is of major concern in the IM program is the structure of the rational number system. What is implied by this concern?

A mathematical system may be defined as consisting "of a set of elements, one or more binary operations defined for these elements, and a relation, together with the properties which the elements are to obey under these operations" (McFarland & Lewis, 1966, p. 280). In the rational number system the elements are the rational numbers, the basic operations are addition and multiplication, the relations include equivalence and order, and the properties include closure, commutativity, associativity, distributivity, and the presence of identity elements and inverse operations. The study of the structure of the rational number system then involves the study of these specific numbers, operations, relations, and properties.

Defining the content of a math program in this way does not eliminate any important part of the content of traditional elementary school arithmetic. Pupils still learn to add, subtract, multiply, and divide; to use the base 10 numeration system; and to work with fractions and decimals. Nor does this emphasis on the structure of the number system mean that there is a continuing emphasis upon a formal study of properties and other components of the system. Rather, the emphasis on structure reveals itself in the way in which many of the traditional operations and other content are presented. For example, rather early in their work on addition and multiplication, pupils solve pairs of simple problems to show that these operations are associative and commutative,

even though these terms will not be used. As another example, an addition table (matrix) might be developed and used not only as an aid in learning the addition facts, but also as the basis for an informal introduction to zero as the identity element in addition, to the inverse relationship between addition and subtraction, and to the basic idea of operations or functions. This largely informal introduction to the structure of number systems should provide an important background to the more formal study of such structures in junior high and in high school algebra.

On the basis of this general description of the basic concepts involved in a definition of a "mathematical system," certain concepts and operations can be identified as providing the major components in the content of the IM curriculum. These include the following:

1. Sets.
2. Number.
3. Addition and Subtraction of Cardinal Numbers.
4. Expressions and "Other Names for Numbers."
5. Radix (Base) Numeration Systems.
6. Multiplication and Division of Cardinal Numbers.

The content associated with each of these topics is described in the following sections of this paper. In each case, the content is introduced by citing definitions and descriptions of the mathematics involved. These definitions and descriptions should be considered as presenting major components of the mathematical content of the program. As such, they provide a basis for the inclusion of related instructional objectives. Also, these definitions are of use to the curriculum developer in providing guidance that will prevent him from developing objectives or learning activities that are not "mathematically correct." Following the descriptions and definitions, there are short discussions of the relevance of these ideas to the curriculum and samples of objectives.

1. Sets

The concept of set is used as a starting point for developing definitions of key concepts in many modern approaches to mathematics. The central importance of the idea of "set" is presented in the introduction of the SMSG program as follows:

> Why sets? We know that most of an elementary mathematics program is concerned with arithmetic, and that this begins with counting and addition. Why start with the idea of sets? . . .
>
> One answer to these questions is that a set is what you count. We count 6 cats, or 6 boys, or 6 ice cream cones. Yet the concept of the number 6 does not depend on cats, boys, or ice cream cones. The thing that is common to all of the collection of objects which we count is that they are sets. The notion of set will recur throughout the entire mathematical program and it is wise to introduce the term early and to use it effectively in building the notion of number. We shall also see that the simplest descriptions of addition and subtraction . . . are in terms of manipulation of sets of physical objects. (School Mathematics Study Group, 1965, p. 1)

Working with sets of physical objects can be an important learning experience as the student is being introduced to mathematics. Levi (1954) states that "a great deal of mathematics can, in fact, be regarded as a logically precise realization of an experience . . ." (p. 18). Sets of objects are used to systematize the experiences of beginning mathematics students in order to provide a framework within which the student learns to describe these experiences in logically precise terms. Although we do not expect the early elementary students to be logically precise in their language, we do expect them to begin to approximate precision.

Descriptions and Definitions

> Set. (a) "A . . . bunch of things" (Kelley, 1970, p. 76); (b) ". . . some collection or aggregate of objects" (Eves & Newsom, 1958, p. 226).

Elements of a set. (a) ". . . the things that belong to the bunch are called members of the set or elements of the set" (Kelley, 1970, p. 76). (b) ". . . the objects which make up a set . . . and . . . we shall put no restriction on the nature of these objects" (Eves & Newsom, 1958, pp. 226-227).

Designation of sets. There are two ways of designating a set: (a) by itemizing each member of the set, i.e., 2, 3, 5, 7, . . . "This is the roster notation" (McFarland & Lewis. 1966, p. 9); (b) by specifying a defining property which every element of the set possesses, i.e., "all the prime numbers less than ten" (Morgan & Paige, 1963, p. 2).

Universal set. (a) "In many discussions, it is useful to regard all sets with which one may be concerned as sub-sets of an over-all embracing set I; I is then known as the universe" (Eves & Newsom, 1958, p. 228). (b) ". . . an overall inclusive set for a particular discussion is called the universal set for that discussion" (McFarland & Lewis, 1966, p. 12).

Pairing. "We have two bunches of things (sets) and we pair off the things (elements) in one bunch with the things in the other. . . . This pairing process decides for us whether there are just exactly as many things in one bunch as in the other or whether there are more in one bunch than the other" (Kelley, 1970, p. 76).

One-to-one correspondence. "Two sets A and B are said to be in one-to-one correspondence when we have a pairing of the elements of A with elements of B such that each element of A corresponds to one and only one element of B and each element of B corresponds to one and only one element of A" (Eves & Newsom, 1958, p. 236).

Equivalence. (a) "Two sets A and B are said to be equivalent, and we write A ~ B, if and only if they can be placed in one-to-one correspondece" (Eves & Newsom, 1958, p. 237). (b) One set matches another, and the two sets are equivalent if and only if it is possible to find a one-to-one correspondence between them" (Kelley, 1970, p. 79).

Comparison of sets. "If A and B are sets, then precisely one of the following occurs: A matches a proper subset of B, A matches B, or B matches a proper subset of A" (Kelley, 1970, p. 83).

Equal sets. (a) "Two equivalent sets are identical if each is a subset of the other . . . we use the symbol '=' " (McFarland & Lewis, 1966, p. 26). (b) "Two sets A and B are said to be equal, and we write A = B, if and only if every element of A is an element of B and every element of B is an element of A" (Eves & Newsom, 1958, p. 227).

Same number. "We return to the notion of pairing . . . which underlies the idea of number. We . . . say that two sets have the same number of members provided the members of the first set can be paired off with the members of the second set so that both sets are used up" (Kelley, 1970, p. 81).

Relevance of Content and Sample Objectives

The ability to pair and to compare two or more sets is basic to an understanding of the concept of number (see definitions of "cardinal number" in following section). Although a majority of children learn to do rote counting before they are exposed to formal instruction in arithmetic, their development of an understanding of number is probably a gradual process, with work on the comparison of sets representing a key aspect of this development. Understanding that sets have the "same number" because they are equivalent when matched one-to-one should also be useful in comprehending number sentences involving the use of the "equal" sign. It can also provide a basis for teaching the concept of the "missing addend" and "other names for numbers." The following objectives represent a sample of IM objectives related to this content area:[2]

Given 2 sets of objects, the student pairs the objects and states if the sets have the same number of objects.

Given 2 sets of objects, the student pairs the objects and states which set has more.

Given 2 sets of objects, the student pairs the objects and states which set has less.

Given 3 sets of objects, the student matches and states which set has the most, least.

---

[2]Objectives listed in this math content section of this paper are intended to serve only as examples of objectives related to given content. A complete listing of all IM objectives, organized into units, is presented in Appendix A.

2. Number

"Number" is a major and central concept in the content of an arithmetic curriculum. In the mathematical systems (such as the natural number system and the rational number system) with which arithmetic is concerned, numbers are the "elements" of the system. The approach taken here to the development of number, as derived from the concept of set, has been explained by Eves and Newsom (1958) as follows:

> We saw how the great bulk of classical mathematics can be reached from the natural number system in a purely definitional way, and thus how the consistency of classical mathematics can be made to rest upon the consistency of the basic number system. It is natural to wonder whether the starting point of the definitional development of classical mathematics cannot perhaps be pushed to an even deeper level. Mathematicians who attempt such a construction usually start with the theory of sets, the concepts of which are already involved in the postulational development of the natural number system. The success of the program enables one to define natural number in terms of sets, and hence to reduce the number of undefined terms that must be assumed in mathematics. (pp. 235-236)

Because of the importance of the child's acquiring a proper comprehension of the idea of number, a systematic and somewhat extended sequence of objectives is needed to include all the necessary concepts. Since the essential process involved in determining the cardinal number associated with a set of a given size is that of counting, this concept and procedure is introduced along with that of number. Also, since "in order to use the concept of number effectively, we have to have names for particular numbers" (Kelley, 1970, p. 86), numerals must be introduced in this same context.

Descriptions and Definitions

> Cardinal number. (a) "Two sets which are equivalent are said to have the same cardinal number. All sets that have the same

cardinal number as the set a are said to have the cardinal number one (or to contain one element); all sets that have the same cardinal number as the set a, b are said to have the cardinal number two (or to contain two elements) . . . and so on. We shall denote the cardinal numbers one, two, three . . . by 1, 2, 3, . . ." (Eves & Newsom, 1958, pp. 237-238). (b) "Bertrand Russell . . . employed the idea of . . . [Definition (a)] in basic fashion when [he] defined the cardinal number of a set S as the set of all sets equivalent to set S" (Eves & Newsom, 1958, pp. 237-238). (c) "The number of members in a set is the bunch of all sets which are equivalent to it" (Kelley, 1970, p. 81).

Whole number. (a) "The set of whole numbers is W = {0, 1, 2, 3, 4, . . .}" (McFarland & Lewis, 1966, p. 108). (b) "The set of natural numbers or counting numbers does not include 0; it begins with 1. The set of whole numbers does include 0. These numbers 0, 1, 2, 3, 4, . . . the 0 is used in the cardinal sense to mean 'not any' and is read zero (as 4 - 4 = 0)" (Flournoy, 1964, p. 18).

Integers. "The set of integers, I, is the union of the set of natural numbers, N, the number 0, and for each $n \in N$, the number $\frown n$ such that $n + \frown n = 0$" (McFarland & Lewis, 1966, p. 157).

Rational numbers. "The set of rational numbers, Q, consists of those numbers which may be expressed in the form $\frac{a}{b}$, a, b, $\in I \neq 0$" (McFarland & Lewis, 1966, p. 209).

Numeral. (a) "In order to use the concept of number effectively, we have to have names for particular numbers. Numbers, like people, may have many names" (Kelley, 1970, p. 86). (b) "The difference between a number and a numeral is precisely the difference between a person's name and the person" (Kelley, 1970, p. 86).

Counting. (a) "We usually say that a set having count n contains n elements or has the cardinal n" (Deskins, 1964, p. 46). (b) "A nonempty set S of distinct objects has the count n, n a natural number, if and only if there exists a one-to-one mapping F of the set S onto the set M, of the natural numbers $m \leq n$. Such a mapping F is a counting of the set S" (Deskins, 1964, p. 46). (c) "The process of assigning the number N(A) to each set A can be called counting" (Kelley, 1970, p. 81). (d) "Invariance property: suppose that A matches a proper subset of B and suppose A* matches A and B* matches B, then A* has fewer members than B*" (Kelley, 1970, p. 83). (e) "If a and b are any two natural numbers, then one and only one of the following situations obtains: $a = b$, $a < b$, $a > b$" (Eves & Newsom, 1958, p. 198).

Relevance of Materials and Sample Objectives

While some work with naming equivalence classes of sets can be done without counting, there is a limit as to how many objects one can recognize visually without counting. The following objectives represent the types that were defined to establish a meaningful basis for counting and for actually teaching counting:

Given 2 objects of different sizes, the student selects the larger (smaller) object.

Given a set of 0-5 moveable objects, the student counts objects moving them out of set as he counts.

Given a set of 0-10 moveable objects, the student counts objects.

Given 2 sets of objects, the student counts and states which set has more (less) objects or that the sets have the same number of objects.

Given 3 sets of objects, the student counts and states which set has the most (least).

Given a set of objects and a set of numerals, the student selects the numeral that represents the set, one more than the set, and one less than the set.

Given a number stated (to 5) and a set of objects, the student counts out subset as stated.

Given a number stated (to 10) and a set of objects, the student counts out subset as stated.

3. Addition and Subtraction of Cardinal Numbers

When a student has acquired some understanding of the concepts set and number, he has been introduced to the "elements" of a mathematical system and is prepared to undertake some work with "operations."

The first formal operation with numbers that the student is introduced to is that of addition. Work on addition and subtraction is preceded by activities with sets, including the forming of unions of sets, the removal of subsets, and "adding to" or "taking away" from one set to make it equivalent to another set. The basic mathematical terms and expressions that are of relevance here are subset, union of sets, difference of sets, intersection of sets, complement of a set, and the definitions of addition and subtraction.

Descriptions and Definitions

Set properties. (a) Subset: "A set T is a subset of a set S ($T \subseteq S$) if and only if every element of T is also an element of S (Deskins, 1964, p. 3). (b) Union of sets: "The union of two sets, A and B, written as $A \cup B$, is the set $x \mid x \in A$ or $x \in B$" (Herstein, 1964, p. 2). (c) Difference of sets: "Given the two sets, A, B, then the difference set, $(A \sim B)$ is the set $\{x \in A \mid x \notin B\}$" (Herstein, 1964, p. 5). (d) Intersection of sets: "The intersection of two sets A and B, written as $A \cap B$, is the set $\{x \mid x \in A$ and $x \in B\}$" (Herstein, 1964, p. 3). (e) Complement: "The complement of a set A relative to a universe I is the set of all elements of I which are not elements of A. When the universe I is clearly understood, we denote the complement of A by A' " (Eves & Newsom, 1958, p. 228). (f) Disjoint: "[When two sets] . . . have no elements in common, that is, their intersection is the empty set, they are called disjoint sets" (McFarland & Lewis, 1966, p. 18).

Addition. "The addition of the whole numbers a and b is the assignment of the whole number c to the ordered pair (a, b) such that $N(A) + N(B) = N(A \cup B)$ where A and B are finite sets, $A \cap B = \phi$ and $N(A) = a$, $N(B) = b$, and $N(A \cup B) = c$. That is, $a + b = c$" (McFarland & Lewis, 1966, p. 108).

Subtraction. "Definition of subtraction using removal of a subset: The subtraction of the whole numbers a and b ($b \leq a$) is the assignment of the whole number c to the ordered pair (a, b) such that $N(A) - N(B) = N(A - B)$ where A and B are finite sets, $B \subset A$, and $N(A) = a$, $N(B) = b$, and $N(A - B) = c$. That is $a - b = c$" (Hosticka, 1973, p. 60).

Relevance of the Material and Sample Ojbectives

The foregoing terms and definitions provide the important mathematical content basis that must be considered in developing curriculum objectives in the area of addition and subtraction. They represent a major basis for the content of the IM program. However, this does not mean that these formal terms and definitions are to be mastered by the student. Instead, they are taken as describing some of the actual physical operations with sets that the student should experience before being introduced to the formal operations of addition and subtraction and which he should be able to relate to these parallel operations with numbers. In the interest of simplication and the elimination of some complications that are not necessary at this level, certain restrictions are placed on the way in which these concepts are used. As an example, although union of sets is important in the introduction of addition, the union must always be of disjoint sets. Since the development of addition is of primary importance, the union of nondisjoint sets is not dealt with at this time.

The following are examples of objectives concerned with helping the student acquire the basic meaning of addition and subtraction of whole numbers:

> Given two sets, the student makes the two sets equivalent by adding or removing objects.
>
> Given two numbers (whose sum is to 9) and a set of objects with the directions to add, the student adds the numbers by counting out two subsets, then combining and stating the combined number as the sum.
>
> Given two numbers (to 9) and a set of objects with directions to subtract, the student counts out smaller subsets from larger sets and states remainder.

Given two sets, the student draws a picture showing the union of these sets. (Limit: Maximum of 9 elements in the union set.)

Given a set, the student shows (i.e., by crossing out objects) the removal of a subset.

After the student has acquired an understanding of the basic operations involved in addition and subtraction, he is ready to learn to read and write number sentences. The number sentence can be generated from the student's work with sets.

Given a representation of union of sets or removal of a subset, the student writes the number sentence. (Limit: Sums through 5.)

Given a representation of union of sets or removal of a subset, the student writes the number sentence. (Limit: Sums from 6-9.)

Given pictures, the student illustrates union of sets or removal of a subset, and writes the number sentence. (Limit: Sums through 9.)

4. Expressions and "Other Names for Numbers"

Expressions are an important part of a mathematics curriculum in that they give the child insights into the relationship of operations and number names. Being able to use an expression as a legitimate name for a number refines the child's concepts of equality and the operation represented in the expression.

Descriptions and Definitions

Expressions. (a) "Numbers, like people, may have many names" (Kelley, 1970, p. 86). (b) "We define the useful concept expression by listing the things that are to be so described. (a) Each cardinal number in standard form is an expression. (b) Each

variable whose domain is the set of cardinal numbers is an expression; (c) If A and B are expressions, then (A) + (B) is an expression; (d) If A and B are expressions, then (A) · (B) is an expression.

Note: Items c and d state that any two expressions can be used to form new expressions by enclosing each in parentheses and then combining them by either addition or multiplication. We attach a further stipulation that if either of the expressions so combined is a cardinal number in standard form or a variable whose domain is the set of all cardinal numbers, then no parentheses are to enclose that expression.

Examples: Items c and d certainly certify as expressions each of the following: $3 + 5$, $3 \cdot 5$, $3 + x$, $x + y$, xy. They also certify as expressions more complicated combinations such as $(3 + 5) + (6 + 11)$, $(3 + x) + (yz)$, and $((x + y)(5z)) + (3w)$" (Levi, 1954, p. 34).

Relevance of Material and Sample Objectives

The importance of attending to the use of expressions as other names for numbers is revealed in the number of children who do not pay attention to signs or understand "equals." A child who reads the number sentence $8 = 2 + 6$ as "eight plus two equals six" not only is showing a lack of attention to signs, but is showing a lack of comprehension of addition and equality. Errors of this type are not uncommon in the elementary grades. In an effort to minimize misunderstandings of this sort, the IM program provides for extended work in writing expressions, equations, and "other names for numbers." Some examples of objectives in this area are the following:

Given a number sentence in the form $a + \square = c$ or $\square + b = c$, the student completes the sentence. (Limit: Sums through 9.)

Given a number sentence in the form $a = \square + \triangle$, the student gives several solutions for the sentence. (Limit: Sums through 9.)

Given a set or a number line to represent a sum, the student writes several addition and subtraction sentences for each sum.

Given a set of objects, the student groups the objects by tens and completes statements of the form _____ tens and _____ ones. (Limit: 1 through 99.)

Given a number written as a standard numeral, the student completes statements of the form _____ tens and _____ ones. (Limit: 1 through 99.)

5. Radix (Base) Numeration Systems

After the beginning student has been introduced to the elementary ideas of number, to the addition and subtraction operations on numbers, and to expressions, the magnitude of the numbers with which the student can work can be increased. This means that the decimal notation system, as the relevant exemplification of the radix numeration system must be introduced. Radix numeration systems are "conceptually, an operational procedure. By counting . . . repeatedly, we can assign to each set of objects . . . (a) string of digits which is the . . . name of the number of objects" (Kelley, 1970, p. 88). The following principles govern the process of numeration used by radix or base systems of numeration.

Descriptions and Definitions

Grouping. (a) "Axiom on counting: Suppose that n is a counting number and that d is a counting number. . . . Then each set of n members can be split into a collection of sets, each having d members, and a remainder set which has fewer than d members" (Kelley, 1970, p. 88). (b) " If a and b are integers and $a \neq 0$, then there exist unique integers q and r such that $b = aq + r$ and $0 \leq r < a$. The element q is the quotient and r is the remainder" (Deskins, 1964, p. 82).

Positional notation. (a) "Let a and b be positive integers, $a > 1$. Then b is expressible uniquely in the radix form with base a as

$$b = a^n c_n + \ldots + ac_1 + c_0$$

where c_n is a positive integer and the c_i are integers with $0 \leq c_i < a$ for $0 \leq i \leq n$" (Deskins, 1964, p. 84). (b) ". . . the radix representation of b, relative to the base a is unique. This result enables us to construct a system of symbols for the positive integers in the following way. For the finite set of non-negative integers less then base a, arbitrary symbols are selected (i.e., the digits) . . ." (Deskins, 1964, p. 86).

Relevance of Materials and Sample Objectives

These mathematical definitions are given in terms of radix representations, which describe the generalized base positional notation. Our base ten Hindu-Arabic system is a specific example of this type of notation. Using these definitions and theorems, our base ten numeration system can be developed and explained. Examples of objectives that develop the pupil's comprehension of base ten numeration and place value are:

Given a standard numeral ≤ 100 and objects, the student counts out and arranges into sets of tens and extra ones the number of objects indicated by the numeral.

Given a standard numeral, the student completes statements of the form _____ tens and _____ ones. (Limit: 1 through 99.)

Given a standard numeral for a number to 999, the student writes numerals to 999 in columns for hundreds, tens, and ones according to the place value of each digit.

6. Multiplication and Division of Cardinal Numbers

Introducing the concepts of multiplication and division in the IM curriculum for the primary grades represents an expansion on the content of the rational number system as a "mathematical structure" in

that it involves additional basic binary operations. Among other things, these new operations can be contrasted and compared with the operations of addition and subtraction that are introduced earlier. This comparison, pursued on a relatively informal basis, can be made in terms of both the processes involved and the properties of the operations.

In developing objectives and lesson materials in this area, it is essential to avoid the oversimplication represented by the idea that multiplication is nothing more than repeated addition. Dienes and Golding (1966) address themselves to this point:

> It is important to realize that in this operation we have gone beyond the idea of addition. It is true that the same answer can be obtained to the problem by an addition of the three terms as by multiplication by three. Just because the answer is the same does not mean that the operation is the same. Multiplication involves a new kind of variable, namely the multiplier, which counts sets. The multiplier is a property of sets of sets. The multiplicand is a property of sets. So the two factors do not refer to the same universes. In fact, there are no factors in the case of addition because the number of addends does not enter into the problem as a variable. Those teachers who teach that multiplication is nothing but repeated addition are doing a disservice to the education of their children. . . . In multiplication, we are dealing with two different universes at the same time, whereas with addition, we are dealing with the same universe, namely that of sets. Every number refers to sets in addition, whereas in multiplication some refer to sets of sets and others refer to sets. This is a very great difference and the exercise children will have had in dealing with sets of sets and even with sets of sets of sets will serve them in good stead in coming to grips with the problems of multiplication at this stage. (p. 34)

Descriptions and Definitions

> Cartesian product. "The Cartesian product of the two sets P and Q, P x Q, is the set consisting of all the ordered pairs whose first component is an element of P and whose second component is an element of Q" (McFarland & Lewis, 1966, p. 95).

Multiplication. (a) "The multiplication of the whole numbers a and b is the assignment of the whole number c to the ordered pair (a, b) such that N (A) x N (B) = (A x B), where A and B are finite sets, and N (A) = a, N (B) = b and N (A x B) = c. That is, a x b = c" (McFarland & Lewis, 1966, p. 113). (b) "A model for multiplication need not depend on any knowledge of addition. . . . Now if m and n represent any whole numbers, the product m · n is the number of members in the union of m disjoint sets of n members each. The operation for finding that product is multiplication" (Marks, Purdy, & Kinney, 1970, p. 126).

Division. (a) "Division is defined as follows: if p, n, q, and r represent whole numbers and $n \neq 0$ so that $p = (q \times n) + r$, $r < n$, q will be called the quotient and r the remainder when p is divided by n. The operation of determining q and r when p is divided by n is called division" (Marks, Purdy, & Kinney, 1970, p. 128). (b) "Partial division property: If n and d are counting numbers . . . , then there are unique counting numbers q and r such that $r < d$ and $n = (d \times q) + r$" (Kelley, 1970, p. 104).

Relevance of the Material and Sample Objectives

Obviously, beginning students are not introduced to the concept of the Cartesian product as a part of their introduction to multiplication. However, in early work with arrays (x rows with y things in each row), they are being introduced to a format which lends itself to a later intuitive understanding of the Cartesian product (i.e., each x position, or row, contains an instance of a y position, or column).

It is important, when developing pupil understanding of multiplication and division to emphasize their inverse relationship. One way this can be done effectively is by developing paired work sentences for the two operations. If, in multiplication sentences, the multiplication sign is read as "sets of," and if a division sentence is read "how many sets of _____ are in _____?," then the multiplication sentence answers the division sentence. For example, 12 : 3 = _____ is "how many sets of 3 are in 12?" The answer is "4 sets of 3 are in 12."

The following are examples of objectives from the IM program that represent a basic introduction to the concepts of multiplication and division:

Given a set < 10, the student replicates and states "a sets of b are in c."

Given objects (20-100), the student groups by a number < 10 and states how many sets of a are in b.

Given an array, or a number line with equal intervals indicated, the student completes statements of the form: "_____ jumps/rows/sets of _____ in _____. (Limit: One-digit factors.)

Given the number of objects in the array and the number of sets, the student makes the array and completes statements of the form: "There are c sets of _____ in b." (Limit: One-digit factors.)

Given a multiplication table for facts, both of whose factors < 5, the student writes the numeral which belongs in each empty cell and uses the table to complete multiplication sentences of the form a x b = □.

Given a multiplication table for facts with factors of 6-9, the student writes the numeral which belongs in each empty cell and uses the table to complete multiplication sentences of the form a x b = □.

Given an array, the student writes division sentences for the array. (Limit: Single-digit factors.)

Summary

The material presented in the foregoing outline and discussion of the mathematical content underlying the IM curriculum should be considered as providing major criteria for selecting and editing objectives, test items, and lesson materials for the program. The presentation of

this content at this point is not intended to imply that the next step in curriculum development was to take this outline and derive each specific instructional objective from these definitions and descriptions. In many ways, this latter step has already been carried out by the SMSG and similar groups of mathematicians and educators who have devoted many years to the specification of the content and objectives of primary grade mathematics. Their work simplified the task of the IM staff. However, the mathematics content outlined here provided a ready reference for the IM staff in checking the correctness and completeness of their objectives, tests, units of instruction, and lesson materials. In this way, it provided criteria for the staff's judgment of their work.

THE PLAN AND STRUCTURE OF THE MANAGEMENT SYSTEM OF THE IM PROGRAM

The IM program consists of a number of components, each one planned and developed to be a part of an instructional system designed to achieve the program goals. These components include the following:

1. The Structure and Sequence of Objectives and Units.
2. The Testing Program.
3. Lesson Materials.
4. Prescription Development Procedures.
5. Classroom Management Procedures.
6. Pupil Classroom Activities.

The Structure and Sequence of Objectives and Units

The IM program, because it built on its two predecessor LRDC math programs and because it was designed to provide for pupil transition to the IPI program at the intermediate grade (fourth-grade) level, incorporated many of the PEP and IPI objectives into its overall structure. However, all such objectives were screened, modified, and supplemented in terms of how they fitted into a modern math program and

were compatible with the mathematics content rationale developed for the project. The result of this process is that some IM objectives are identical with those found in PEP or IPI, some are modifications of objectives from those programs, and some are new objectives developed for the IM program.

The IM objectives are organized into units, as are those of both PEP and IPI. The relationship among units in IM is a hierarchical relationship, an extension of the type of hierarchy found in PEP Quantification. However, since the 45 units in IM cover a much wider range of topics than the 14 units in PEP Quantification, the hierarchy relating the IM units is considerably more complex than that of its predecessor. This 45-unit hierarchy is presented in Figure 1. As suggested earlier, the possibility of developing and using a hierarchy of this degree of complexity was one of the general curriculum development questions examined through the experience with the IM program.

The IM unit hierarchy is to be interpreted in the manner appropriate for other such structures. This means that for any given unit, the units that are prerequisite to it are below it in the curriculum and tied to it by a solid line. In program operation, the principal use of this hierarchy is to suggest various alternative sequences that pupils can follow in moving from unit to unit. This permits the selection of a path through the curriculum that seems particularly adapted to the needs and interests of each individual student. It also gives the teacher guidance in preparing alternative prescriptions in situations where a pupil encounters major difficulties with some one unit or where all the manipulative exercises in a unit are already in use.

Each IM unit is made up of several objectives. An objective can be considered the basic instructional element in the curriculum. Each objective describes a specific mathematical ability that the pupil should acquire. For each objective there are specific lesson materials (booklets, manipulatives) that have been designed to teach that objective and

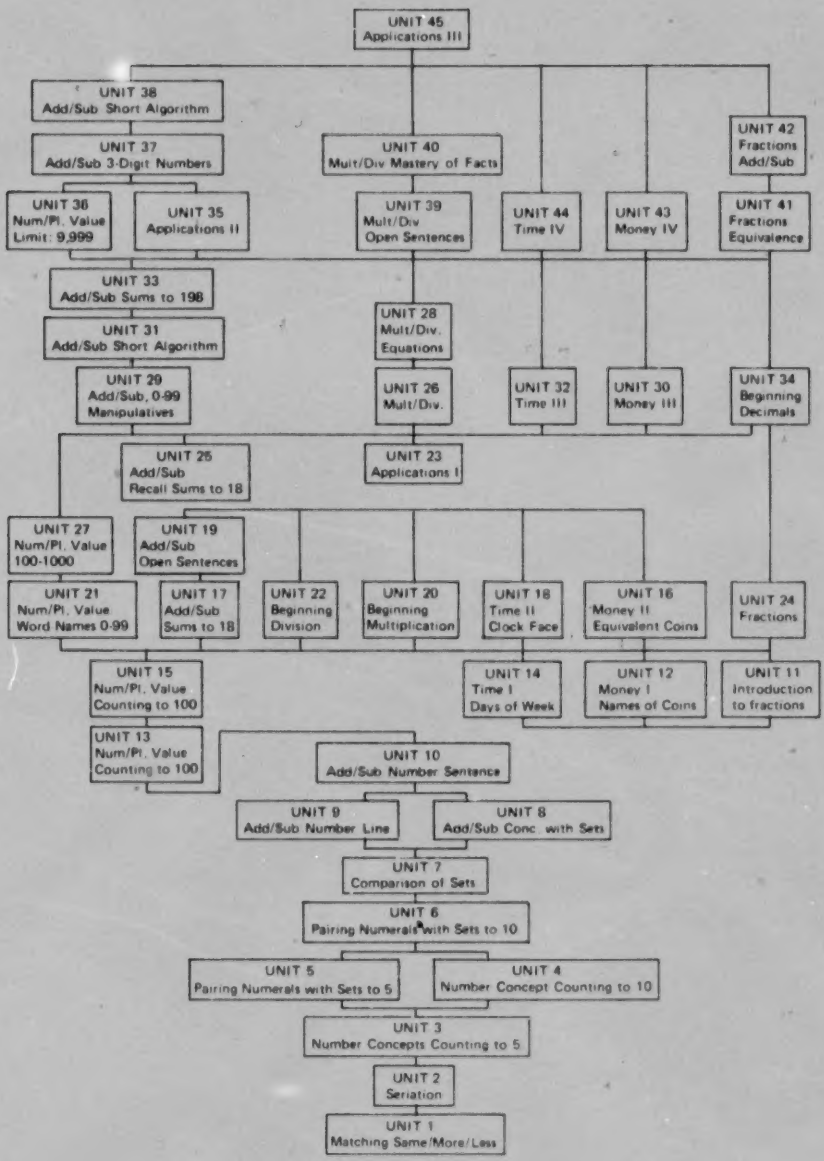

Figure 1. Hierarchy of the 45 units in the IM curriculum.

which the pupil can use largely on an independent study basis. Also, the system includes a test (CET) for each objective that is used for determining when a pupil has mastered that objective and is ready to move on to the next one. In the IM program, the objectives within a unit are organized into hierarchies which indicate the prerequisite relationships involved. An example of one such unit hierarchy is shown in Figure 2.

Note that each objective within the unit is identified by a letter and the statement of the objective tells what the stimulus situation (the "given") is to be and what specific behavior represents the correct response. In all such hierarchies of objectives within a given unit, the prerequisite ordering is from the bottom up. In this unit, Objective A is prerequisite to B, which in turn is prerequisite to C. C is a prerequisite to both E and G. D is also prerequisite to E, and the latter objective is prerequisite to F. In the labeling of booklets and of tests for a given objective, both the unit number and the letter designation of the objective are used. The objectives in this unit would have such labels as M4A, M4B, M4C, and so on (the M merely identifies it as a "Math" objective).

The Testing Program

As is true of the IPI and PEP programs, the IM program uses a procedure for systematic testing as a major tool for facilitating individualization. All tests used in the program are criterion-referenced tests in that they provide information concerning what the pupil has or has not mastered.

Placement Testing

Placement tests can be used at the start of the school year or at any time a new student enters the program in order to determine where the student should start studying within the hierarchy of 45 units. The

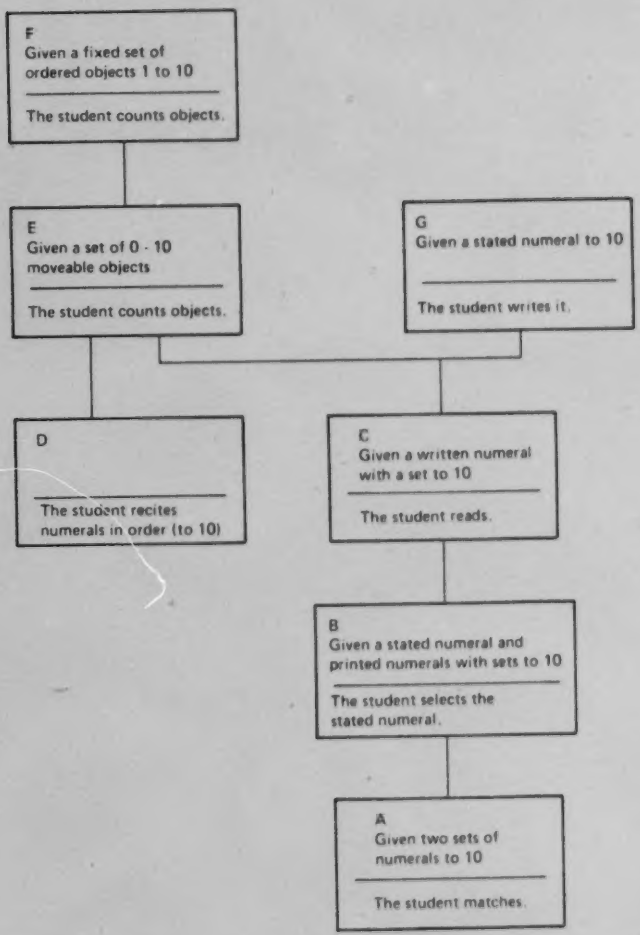
F
Given a fixed set of ordered objects 1 to 10
The student counts objects.
E
Given a set of 0 - 10 moveable objects
The student counts objects.
G
Given a stated numeral to 10
The student writes it.
D
The student recites numerals in order (to 10)
C
Given a written numeral with a set to 10
The student reads.
B
Given a stated numeral and printed numerals with sets to 10
The student selects the stated numeral.
A
Given two sets of numerals to 10
The student matches.

Figure 2. Unit hierarchy for IM Unit 4.

total battery of placement tests for the IM program consists of individual tests for each unit in the program. In the interest of efficient testing, the exact units on which a pupil is placement tested are determined by past performance in the program and by the teacher's judgment. For example, most students entering first grade are given a placement test package including tests for Units 1 through 7. This provides a sufficient basis for placement decisions for most students. Placement testing at higher grade levels is done by assembling a package of tests covering those units judged most appropriate for a particular class or even a particular student.

Unit Pretesting

When a student is to begin work in a given unit, he first takes the unit pretest. This test provides a score for each objective in the unit, and such scores provide information as to whether there are any objectives that the pupil does not need to study. On the basis of pretest results, a decision is made as to which objective the pupil should study first. The exact nature of the results from pretesting can perhaps best be seen by examining the form used to record a student's scores on both pretests and posttests for one unit. The form shown in Figure 3 is for Unit 26 in the IM program.

Skill	Points	Pre	%	Post	%	Post	%	Post	%
A	8								
B	5								
C	6								
D	6								
E	8								
F	4								
Date									

Figure 3. Form for recording a pupil's pre- and posttest scores on each skill in a given unit.

As can be seen, there are six objectives or skills in this unit, and each one is tested by some limited number of items (8, 5, 6, etc.). If a pupil shows mastery (i.e., has all items correct) of a skill on the pretest, he will not be prescribed study in that skill.

Curriculum-Embedded Testing (CETs)

When a pupil completes his study of a given skill or objective within a unit, he takes a CET to determine if he has mastered this skill. For most skills in the IM program, the CET is a paper-and-pencil test appearing at the end of the lesson booklet. For a few skills, such a paper-and-pencil test is not appropriate, and the decision on mastery of the objective is made by the teacher on the basis of pupil performance with the manipulative lesson materials.

Unit Posttesting

When a pupil has mastered all of the objectives in a unit, he is ready for the unit posttest. The posttest is essentially a parallel form of the pretest and, as can be seen by the record form for Unit 26 (Figure 3), yields a score for each skill. If a student does not master a given skill on this test, he is prescribed further work on it and given a second posttest. When he shows posttest mastery of all skills in a unit, he is ready to move on to the next unit.

Lesson Materials

Major components of the IM program are the various instructional materials provided for pupil and teacher. These can be categorized as follows:

A. Lesson booklets.

B. Boxes (containing manipulative lessons).

C. Laminated booklets.

D. Seminar materials.

E. Maintenance materials.

F. Math Lab materials.

Lesson Booklets

The IM program, on the basis of the Center's experience with IPI, adopted the use of lesson booklets as a proven and valuable procedure for individualized instruction. Many of the booklets used in IM are the published IPI booklets, stamped with the appropriate IM designation. Some of the booklets in IM were developed especially for this new program. Each lesson booklet teaches the skill defined by one objective and is identified by the unit number and objective letter. A typical lesson booklet starts with one or two pages of review, follows with several teaching pages, has one or two summary pages that provide practice on the skill as tested by the CET, has a page or two for CET I, provides additional teaching and summary pages, and then CET II. Suggested procedures for classroom use of such booklets are discussed in a later section of this paper.

Boxes

A major aspect of the IM program is its extensive use of boxes, each one containing a manipulative lesson of the type first used with PEP. Each box teaches the skill, or a part of the skill, defined by one objective. For most objectives where manipulatives are appropriate, more than one box is provided. For this reason, boxes are identified by the unit number, objective letter, and box number. For example, Objective B in Unit 13 has three boxes, designated as M13B1, M13B2, and M12B3. Directions for using the material in a box are found on the inside of the top cover. The directions include a picture of how the student is to arrange the manipulative materials for one exercise. This picture is the "model," and the pupil knows that his first step in using

a box is to duplicate the model. When the teacher checks and sees that this has been done correctly, the student is told to proceed by forming a comparable arrangement of materials for each of the other exercises in the box. Successful completion of this task is followed by the teacher checking the work and asking a few appropriate questions.

Laminated Booklets

In a minor revision of the IM program, now being given a rather intensive trial, a number of boxes have been converted to laminated booklets. This was done in an effort to conserve on materials and on needed classroom storage space. Boxes which were converted were those containing laminated pages upon which the pupil records an answer after using manipulative materials to derive it. Such boxes typically use manipulatives such as counting cubes, Dienes blocks, number lines, and so on that are used in several boxes. The conversion to laminated booklets means that the manipulatives are kept in "central storage" in the classroom and are obtained by the student when he determines what his particular booklet requires. Each laminated booklet was produced by using plastic binding to form a booklet from several laminated work pages. Such booklets have the same designation as boxes except that "LB" has been added, e.g., M13C3 (LB).

Seminar Materials

The IPI Program includes the suggested use of whole-class "seminars," or large group instruction, on an average of about one day per week. The IM program continues this feature and provides outlines of suggested activities and pupil practice pages for a limited number of key topics. Also, certain topics included in IPI, such as money and time, are represented in the IM program merely by units and objectives and are to be taught in seminars.

Maintenance Materials

A structured maintenance program is another feature of IM that has been largely borrowed from the Oakleaf School version of IPI. The purpose of maintenance is to help the student maintain and to increase proficiency in mastery of the "basic facts" in the operations of addition, subtraction, multiplication, and division. The program is organized into 17 units at 4 levels, representing a progression in difficulty over the four operations. Materials provided for each unit include a criterion test and one or two practice sheets. Suggestions for use of the program anticipate extensive use of other drill materials and games to be identified by the teacher.

Math Lab Materials

Development work on the IM program included some exploratory work on math lab activities covering certain beginning topics in the areas of measurement and geometry. These activities and materials were developed and used under a variety of plans for teacher supervision. This work was carried out in the first and second grades at Oakleaf School. Although the math lab continues as a part of the IM program at Oakleaf, work on this aspect of the program has not reached the stage where it can be considered ready for more widespread implementation.

As suggested by the foregoing summary of lesson materials and procedures used in the IM program, there is considerable variety available to teacher and pupil. This variety permits flexibility in giving pupils prescriptions that meet their learning needs and contribute to regular progress. The variety in lesson material is also designed to facilitate the development of study skills associated with different types of lessons.

Prescription Development Procedures

When a pupil working in the IM program has taken a unit pretest, the results provide a basis for deciding which specific objectives he should study. The pupil's individualized plan of study for mastering a given objective is outlined in the "prescription" for that objective. The prescription lists the specific boxes, lesson booklet pages, games, and so on that the student is to use. In the current version of the IM program, a pupil's prescription is shown by appropriate entries on the "ticket" for the unit. An example of such a ticket is shown in Figure 4. (Marginal notes have been added to indicate what each ticket entry represents.)

Among other things, the ticket tells the teacher and pupil exactly what materials are available for studying each objective. For Objective 17A, for example, there are two boxes of manipulatives (M17A1, M17A2) and a booklet (M17A). For Objective B there is one laminated booklet and a booklet, while for Objective C there is only a booklet. The row of cells to the right of the symbol for each box or booklet provides spaces for indicating what the particular student is to study and for keeping a record of his progress. (Specific procedures for marking a ticket are described in the Teacher's Manual (Lindvall, Note 1).

To explain the prescription development procedure, it may be useful to consider a hypothetical student. Assume that this student did not show mastery, on the pretest, of any of the three objectives A, B, or C of Unit 17. In this case, the teacher might first give the pupil a prescription for Objective A and must decide what materials are to be prescribed. One possibility would be to prescribe both boxes and the entire booklet. This could be the correct prescription if the student missed all, or most, of the items for this skill on the pretest and, in the teacher's judgment, could not master the basic idea of the objective without working with manipulatives. If, on the other hand, the pupil had narrowly missed the mastery score on the pretest, the teacher could examine the

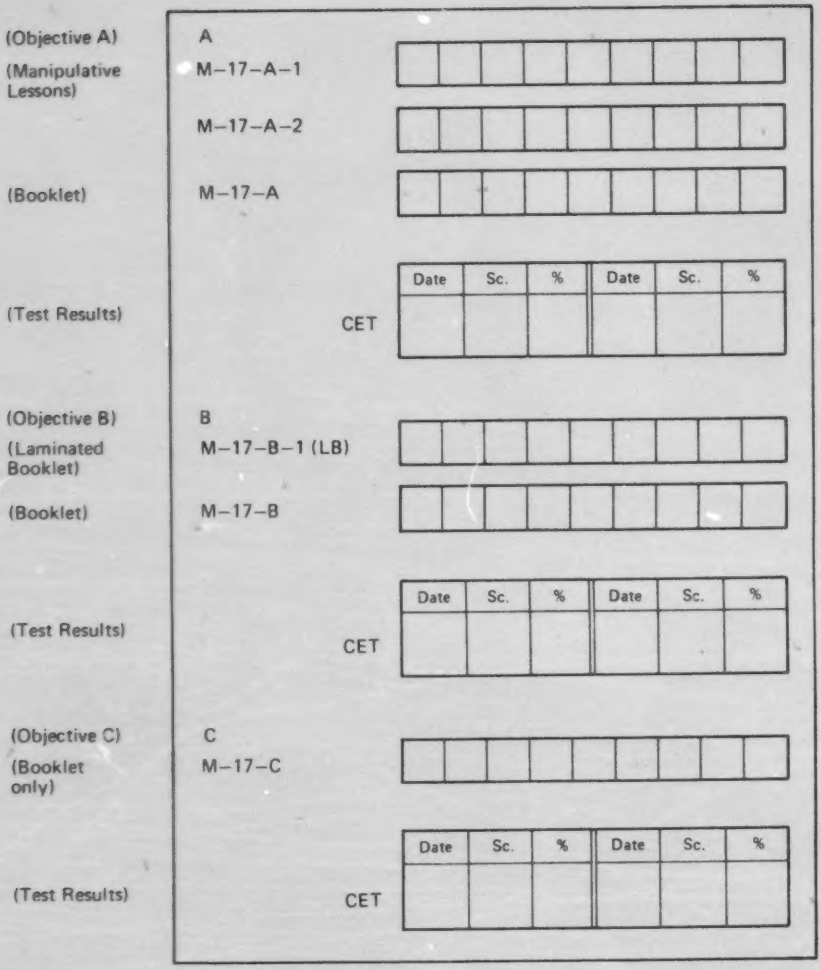

Figure 4. Prescription form for Unit 17 in the IM program.

items that he had missed and prescribe only those pages in the booklet that taught that specific aspect of the skill. For this latter prescription, the teacher could mark the exact pages to be studied on the prescription form found on the front of the booklet. If either of the boxes involved activities designed to teach the specific aspect of the skill in which the pupil was deficient, it too could be prescribed. Quite obviously, other variations in the prescription developed would be appropriate for other situations and other types of students.

In the IM program, prescriptions developed for the lower level units generally make much greater use of manipulatives (boxes) than prescriptions for higher level units. This is a function of the fact that earlier units provide more boxes, which fact, in turn, is largely a result of the fact that skills at these levels are basically manipulative skills. That is, beginning arithmetic is a matter of comparing sets, counting objects, joining sets, removing subsets, and so on. Basically, these activities are most directly experienced and understood through handling concrete materials.

It is the intention of the IM program that paper-and-pencil activities be introduced rather gradually, starting at an early level in the curriculum. As a result, lesson booklets are available for use even in Unit 1. This does not mean that all pupils are to use booklets at this level. Some students may work solely with boxes in the first several units. However, the booklets are available for exploratory use by some pupils and as a part of a regular prescription for others. This provision for a gradual introduction to work in booklets is intended to facilitate each pupil's acquiring increased study skills at a pace suited to his or her capabilities.

Classroom Management Procedures

As is true with any instructional program, the key factor determining the success of the IM program is the skill of the teacher in providing

overall supervision of the program and in interacting with individual pupils. The teacher is provided with all of the material components of the program--tests, lesson booklets, boxes, prescription forms, and so on--but how the system actually operates is dependent upon the teacher. Because of this crucial importance of the teacher role, the IM program has given major attention to this component of the system. This attention is reflected, in part, in the documents that have been used in teacher training sessions. These include the following:

1. Individualized Mathematics Teacher's Manual (Lindvall, Note 1).
2. Individualized Mathematics Testing Manual (Lindvall, Note 2).
3. "Classroom Management of the Individualized Mathematics Program" (Lindvall, Note 3).
4. "Individualized Instruction: Some Suggestions for Teachers and Supervisors" (Lindvall, Note 5).

These documents, taken together, comprise a set of rather detailed suggestions for teacher management of a class using the IM program. The following description of procedures is largely a summary of basic considerations outlined in terms of the type of pupil growth that the program is designed to enhance. (Also, in describing other components of the program in preceding sections of this paper, some aspects of classroom management were necessarily included.)

The Basic Goal

The teacher in the IM program is asked to approach the task of classroom management from the point of view that it is a matter of "managing an individualized instruction situation." This means that the teacher is not to focus attention on such matters as "How am I going to explain this lesson to the class?" or "What kind of activity am I going to conduct today?" Instead, in the individualized classroom the basic

question of the teacher should be "How can conditions be arranged so that each pupil progresses at an appropriate pace, using instructional materials that best facilitate progress?" Although some teacher tutoring and guidance will be essential, the basic goal of classroom management is to help the pupil become a confident self-directed learner capable of planning and pursuing his or her classroom work with a high degree of independence.

Promoting pupil self-management. A successful system for individualized instruction almost necessarily requires that each pupil assume some responsibility for the management of his or her own instruction. Also, of course, the ability to assume this responsibility should be an important outcome of this type of instruction. One element in the IM program that has been developed to enhance pupil self-management is the pupil prescription. This specifies what the pupil is to study. Consequently, an important aspect of teaching procedure is to see that every pupil has an individual prescription at the start of each class period. It may also require that some prescriptions are developed during the period, as a student completes work on a given lesson. However, every effort should be made to do most prescription writing out of class so that class time can be devoted to other activities. Of course, if prescriptions are to be of value, pupils must be able to use them correctly. They must know what the entries on the ticket mean and where to find the indicated materials. This represents one step toward pupil self-management. At higher grade levels, pupils could go farther and could play a part in developing their own prescriptions. However, in the primary grades, being able to follow a prescription, to find one's own lesson materials, and to progress according to a planned program represent a significant degree of responsibility for one's own activity. This is emphasized in the management of IM classes.

Developing pupil study skills. If the teacher is to be a facilitator of individualized progress on the part of each pupil, he or she must be

concerned with pupil success in using lesson materials. One aspect of this, of course, is the development of the best possible prescription for the student. Another aspect is attention to the pupil's ability to use lesson materials properly. The proper management of the IM program requires that the teacher carefully attend to helping pupils to "learn how to learn" from boxes and from lesson booklets. These materials were designed to be used in a particular way and can be expected to function most effectively if used properly. Acquiring such learning skills is essential to pupil progress and also represents a major goal of the program. Of course, of equal importance to the student's knowing how to study is the desire to apply oneself to study activity. Here, again, teacher classroom management is crucial. The teacher must see to it that studying is a rewarding experience. One important step here is developing prescriptions adapted to each pupil's present needs and capabilities so that success can be achieved. This assumes that success, that is, learning as demonstrated by passing tests, has been made a rewarding experience through regular use of teacher praise or other types of positive attention. Obviously, an even more direct practice for increasing pupil attention to their work on lessons is extensive use of teacher praise for this type of behavior.

Enhancing pupil self-concept. The management of all aspects of the testing program is another key component of the IM system. If pupils are to be successful in pursuing independent learning, they must learn to make valid assessments of their progress. Each pupil should acquire the habit of making regular use of the question "Do I really know this?" and then looking for the correct basis for answering it. To provide maximum help in this task, the IM system uses all of the various tests described previously. If these are to be used properly, both pupil and teacher should think of these tests as sources of information. They are not screening devices for determining who is to pass and who is to fail, nor are they instruments for grading. Because they

provide direct information concerning degree of mastery of specific skills, they provide easily interpretable data for guiding instructional decision making. If these tests are to serve their purpose, they must be used "with respect." Pupil and teacher must recognize that they can be useful only if they are administered in a valid manner, are scored correctly, and if results are properly interpreted.

In the IM program the unit posttest is considered a major tool in the management of the system. It provides measures of the extent to which the pupil has really mastered and has retained the skills taught in the unit. Classroom management should include procedures designed to make such posttests major milestones in the pupil's assessment of his or her progress. If success on the posttest can become an important goal for the student, then study activities, CET's, and the pretest can be viewed as tools for insuring success on the posttest. This means, for example, that a pupil should engage in careful review before taking a posttest. This review should be a standard procedure, and it should be emphasized that it is done so that the pupil can determine if he or she is really ready to take the posttest. One specific type of review could entail the pupil reexamining the pretest and becoming convinced that he or she can work the various types of exercises involved. A stress on unit posttest achievement should also lead to pupil concern for performance on each CET. Does the pupil pass the CET with a real confidence in personal mastery of the skill involved? An aid in achieving this is a careful use of the booklet pages immediately preceding the CET (the summary pages) as a pupil self-test for determining readiness for the CET. As all of this indicates, a major purpose of classroom management is to increase the pupils' abilities and desires to judge their own achievement and progress and to use this judgment in planning study activities.

Pupil Classroom Activities

In designing an instructional system, particularly an individualized system, it is essential that major attention be focused on pupil classroom activities, on those types of things that a student must do if program goals are to be achieved. Other system components, such as the teacher's classroom management procedures, the lesson materials, and so on, will all help to determine what a pupil does. However, whether or not the pupil carries on the proper kinds of activities is such a key criterion for determining successful program operation that it is essential that these activities be specified and that they be a focus for program design and implementation. As outlined in an earlier section of this report, the IM program is designed to achieve certain goals in terms of the abilities and attitudes of students. Effective achievement of these goals requires that pupils carry out certain activities and behave in certain ways in the classroom. These can best be outlined in terms of behaviors pupils should exhibit as they interact with other system components. The following outline indicates some of the basic classroom behaviors that should be of concern:

A. Using prescription forms
   1. The pupil will use the prescription to identify what he or she is to study.
   2. The pupil will secure the proper lesson materials.

B. Using lesson materials
   1. The pupil will give appropriate attention to his work.
   2. The pupil will study quite independently and seek help only when it is needed.
   3. In working with boxes, the pupil will follow the recommended steps:
      (a) remove all materials from box
      (b) duplicate model; obtain teacher approval

(c) apply same procedure to other exercises

(d) have teacher check completed work.

4. In working with booklets, the pupil will employ the recommended procedures:

(a) will study directions and examples

(b) will apply procedure to other exercises

(c) will use summary pages as a self-test for determining readiness for CET.

C. Using tests

1. The pupil will make valid decisions as to when he or she is ready for a CET.
2. The pupil will take all tests with care and in a valid manner.
3. The pupil will review before taking a posttest.
4. The pupil will reexamine a completed test to check the answers before having it scored.

As these items suggest, successful operation of the IM program depends upon the pupil using specific procedures in managing his or her own activities, in using lesson materials, and in taking tests. Basically, these are procedures that facilitate effective independent study and, as such, are essential components of the total IM system. Abilities of this type also represent desired pupil outcomes and are to be emphasized by the teacher in daily supervision of pupil learning activities.

Summary

The IM program, as an instructional system, involves six major components: (a) the structure and sequence of objectives and units, (b) the testing program, (c) lesson materials, (d) prescription development procedures, (e) classroom management procedures, and (f) pupil classroom activities. The first three components involve materials produced by the project staff and placed in the hands of teachers and

pupils for their use. The last three components involve procedures and activities outlined by the developers as preferred procedures for managing classroom learning. The outcomes produced by the IM program depend upon the quality of the materials provided and upon the way in which classroom procedures and activities are carried out. Current implementations of IM are operating in two schools in the Pittsburgh area where project staff members have played a role in teacher workshops, in providing consultative help to teachers, and in other activities designed to assist in the effective implementation of procedures and activities. The evaluation efforts and results, described in the next section of this report, are for these schools. However, as is discussed in the final section of the report, any specific implementation of a program such as IM is a type of further development of the program, a development which adjusts the program to the environment and to the needs of the local school situation. This local implementation and development will determine the types of outcomes achieved through the use of IM. The nature of some possible outcomes is described in the final pages of the evaluation section.

THE ROLE OF EVALUATION IN THE DEVELOPMENT OF THE IM PROGRAM

Up to this point, most of the evaluation activity associated with the IM program has been formative evaluation, that is, evaluation carried out for the purpose of improving the development effort. This program of formative evaluation employed the general rationale used in the development of IPI Math (Lindvall & Cox, 1970). Following this rationale meant that the development process emphasized four major steps: (a) specifying the goals to be achieved, (b) outlining a detailed plan for achieving these goals, (c) putting the plan in operation, and (d) assessing the extent to which the program, when correctly implemented, achieves its goals. Each of these steps was accompanied by its own type of formative evaluation.

Evaluating the Worth of Program Goals

The evaluation of the goals themselves was largely a matter of defending their worth in terms of (a) their being an outgrowth of the LRDC's long-range plans for the improvement of education, and (b) their being in line with what mathematics educators and child development specialists deem to be important outcomes of instruction in the primary grades. This type of rationale has been presented in some detail in earlier sections of this paper and in the initial planning document "The LRDC Mathematics Project" (Hosticka et al., Note 4).

Evaluating the Program Plan

The evaluation of the plan, as a general guide for the proposed development work, had to be in terms of its potential for achieving the goals. As explained earlier in this report, much of the basic plan of the IM program involved taking the best components of PEP and IPI and incorporating them into the new program. As a result, an important source of evaluation "data" concerning the potential value of the IM plan was the record of the past success of these components as they were used in PEP and IPI. Another source of information here was feedback from teachers concerning their judgments as to the probable effectiveness of various parts of the plan. To obtain this, regular sessions were held with teachers who were experienced in the use of the predecessor programs. In these sessions, these teachers were asked to examine proposed new objectives and units of study, plans for new lesson materials, and possible new classroom procedures. Feedback from these sessions had a major impact on the plan of the IM program.

Evaluating the Program in Operation

Of course, the most extensive and most important type of formative evaluation used in conjunction with the development of IM was that carried out as the various parts of the program were put into operation

in the classroom. When a new program is put into operation in its initial tryout in an actual school situation, the concerns of the evaluator can be outlined in terms of certain basic questions:

1. Is the program operating in the manner described in the program plan? Before information can be gathered as to the effectiveness of the program and its underlying rationale, the necessary time must be taken to insure that the operating program is a true exemplification of the planned program. This means that a major task of the formative evaluator at this point is to gather all the types of information necessary either to certify that the program is operating in accordance with the plan or to alert the developers and implementers to those specific aspects of the operation that do not meet this criterion.

2. If the program is not operating in accord with the plan, what steps should be taken to solve this problem? At first glance, it would appear that the obvious answer to this question is to change the operation to fit the plan. On occasion, the answer is that simple. More frequently, the evaluator faces the need for obtaining additional information in order to determine exactly what needs to be changed. In a program involving many interdependent components, a difficulty with one component may be the result of malfunctioning in one or more other components. In some cases, a quite different response to the above question may be necessary. It may be that the particular aspect of program operation cannot be, or should not be, made to conform to the plan. That is, information obtained during this initial operation of the program may force the decision that the plan itself can be made more useful through certain modifications.

3. Is the program operating according to the planned procedure but not achieving the program goals? A basic purpose of program tryout is to obtain information that can result in program improvement. To serve this purpose, data must be gathered to determine whether the operating program achieves its goals. If it offers no promise for

achieving the desired results, it must be modified. Note that this involves a modification in the plan of the program as well as in its operation. It is to be emphasized that such a modification should not be undertaken until the developer is satisfied that the originally planned program has been shown to be inadequate.

These three questions were the basic concerns in the evaluation of the initial implementation of the IM program. The developers and evaluators were concerned with such questions as: "Are the lesson materials teaching as planned?" "Are all tests functioning properly?" "Are classroom management procedures being implemented as planned?" These and many similar questions provided the focus for the evaluation of the program operation. All of the procedures used in this phase of evaluation cannot be described in this report, but some description of what was involved can be outlined in terms of sources of information used.

Information Sources

Classroom observation (participant observation). An essential first focus of evaluation of the operation was in attending to the specific problems encountered by pupils and teachers as they used the new lessons, tests, record forms, and other similar materials. To accomplish this, provision was made for having a member of the development staff in the schools and classrooms on an essentially daily basis. This person (and frequently more than one person) observed classroom operation, worked with pupils on lessons, discussed problems with teachers, and endeavored to be responsive to any type of feedback that indicated a need for program modification. This provision for having a developer/evaluator present in classrooms continued for approximately two years. Information obtained through this procedure was personally transmitted to other members of the development staff who used it as a basis for program modification.

Use of test results. An obvious and important type of data used in formative evaluation was pupil performance on CETs for each of the many lessons. Lessons for which any significant number of pupils could not pass a first CET were subjected to further analysis (including discussions with teachers and pupils) in an effort to identify and remedy any problems. A similar use was made of results from unit posttests.

It may be worth noting that a program such as IM (as well as IPI and PEP), which uses CETs and unit posttests that are criterion-referenced to specified objectives and that are also "referenced" to specific lesson materials, have a type of built-in formative evaluation. Since these tests are given regularly as a part of program operation, data concerning typical pupil performance on each lesson are rather automatically collected. Such information almost forces itself on the attention of the teacher or other responsible persons and, if it is disappointing, leads to an investigation of causes. Although most systems of instruction make some use of pupil tests, the use of results for systematic formative evaluation is far from typical. In the IM system, the fact that gathering such test data is a regular part of program operation and the ease with which results can be associated with specific program components should enhance the chances that systematic formative evaluation will be employed.

The formative evaluation of the IM program also made use of other types of test results. Since an IM placement test, which provides a measure of pupil mastery for each unit in the curriculum, was available, it was employed to obtain a measure of pupil retention of skills that they had mastered. Feedback from such testing resulted in suggestions for program modification, such as changes in criteria for certifying initial mastery of a unit whether this was done on the basis of unit pretest or unit posttest results.

Feedback from teachers. Another important source of evaluative feedback concerning the operating program was meetings with teachers,

whether the feedback was an incidental output from a regular meeting or an intended outcome of a special meeting called for the express purpose of discussing the IM program.

"Unobtrusive measures." Having a member of the development staff working in classrooms on a daily basis also permitted the gathering of a variety of types of data, some of it of the "unobtrusive measure" type (see Webb, Campbell, Schwartz, & Sechrest, 1966). For example, at times it became evident that certain boxes were not becoming soiled or worn to the same extent as the typical box. This evidence of lack of use became the signal for an investigation of why this was true. A similar signal was obtained when it was noted that the supply of lesson booklets for a particular objective was not being depleted at the expected rate. Further information of the same general type was obtained when unit posttests for a particular unit were not being used up. All such indications of a lack of use of particular materials had to be followed up to determine the reasons for the situation. In some cases, materials were inadequate, a test might seem invalid, or a unit might appear out of place in terms of the prerequisites it demanded. Formative evaluation of this type was a sequential process. When evidence of inadequate operation in some part of the program was obtained, a list of possible explanations for this breakdown was formulated. Next, each of these explanations was investigated to determine its plausibility. In most instances, this investigation involved changing some given condition (revising a box activity or a booklet page, changing prescribing procedures, modifying a test, changing a classroom procedure) and noting whether this corrected the problem.

Pupil records. A major source of formative evaluation data was the record of pupil progress. Such records were scanned regularly to note such things as the amount of time pupils were spending on given objectives and given units, whether certain units or objectives were being skipped, and the amount of retesting required in units.

Assessing the Achievement of Program Goals

The extent to which the IM program has achieved its development goals at its present stage of development can be examined most meaningfully in relationship to the four process goals which were described on pages 4-5 above. In essence, these goals describe the important characteristics of the program that were to be developed. An important aspect of evaluation at this point in development is to ask the question "Does the program exemplify these characteristics?" More ultimate questions concerning the effects of this type of program on students can only be partially answered at this point. Final answers can only be obtained when the development work associated with a complete implementation of the program in classrooms has been accomplished.

Process Goal 1

The curriculum will be based on mathematics content which emphasizes an understanding of mathematical systems as well as competency in performing operations. The description of the mathematical content of the IM program provided in an earlier section of this report indicates some of the steps followed in working toward this goal. That section describes the "mathematical systems" orientation and the specific content, in the form of definitions and principles, derived from this orientation. It also gives examples of specific IM objectives related to each such definition or principle. The concern for competency in performing arithmetic operations is reflected in specific units and lessons that teach these operations and also in the Maintenance program which gives the student extra drill and practice on such skills. An evaluation of the extent to which the characteristics described in process Goal 1 have been built into the IM system can best be made by examining the content of the program and by analyzing the rationale for that content as presented in earlier sections of this paper.

Process Goal 2

The specific objectives of the curriculum will be organized into hierarchies within units, and units will be organized in a structure of hierarchical relationships in such a way as to permit identification of prerequisite relationships and alternative instructional paths. The fact that the units and the objectives in IM were organized in these types of hierarchies is attested to, at a basic level, by the presence of the hierarchy among units as shown in Figure 1 of this report and by the 45 different unit hierarchies found in the IM Teacher's Manual (Lindvall, Note 1). The extent to which such hierarchies identify "prerequisite relationships" has not been totally validated up to this time. Some studies, using empirical data to investigate such prerequisite orderings, have been carried out and have resulted in certain changes in hierarchies (DiCostanzo, 1974). However, additional studies, some currently underway, have yet to be carried out. The fact that these hierarchies "permit identification of . . . alternative instructional paths" has been verified by informal observation of the fact that many teachers do indeed use the hierarchies for this purpose.

Process Goal 3

The program will provide for some variety in instructional materials, including paper-and-pencil and manipulative activities, and provide for the progressive development of independent study skills. The first part of this goal has been achieved. As the description found in an earlier section of this report has indicated, lesson materials developed and found usable for the IM program include booklets, boxes of manipulatives, games, laminated booklets, maintenance drill materials, and outlines and practice pages for use in group instruction. Quite obviously, as further efforts are made to meet the learning needs of individual students, additonal types of lessons may be added and those currently being used should be improved. This is a major goal of work being planned for the

next two years. However, the variety in materials now available in the IM program represents a definite forward step from previous LRDC efforts. The "progressive development of independent study skills" has been built in as a definite part of the IM curriculum. Procedures for achieving this have been outlined in supporting documents on classroom management (see Note 3 and Note 5).

Process Goal 4

The program will be an instructional system made up of specified components involving materials and specific procedures for pupil and teacher use of these materials. As explained in other sections of this report, a major purpose in developing IM was to "systematize" many components and procedures initiated by PEP and IPI. Experience with these earlier programs indicated that it was not sufficient to suggest merely that teachers do certain things (e.g., "use manipulatives," "prescribe alternative paths through the curriculum," "teach study skills," etc.). It was the responsibility of the program developer to build things into the system that made these steps possible. This meant that the total system had to include more preplanned components. This goal was achieved in the IM program by giving attention to such components as (a) manipulative lessons with models for pupil duplication and suggested questions for use by the teacher, (b) instructional hierarchies as guides to prescribing, (c) a teacher's manual with detailed suggestions for use of all materials, (d) á guide for use in classroom management, and (e) outlines of seminars on specific key topics.

Some Descriptive Evaluation Data

The development and tryout of a new instructional program, particularly in a basic skills content area such as mathematics, carries with it a responsibility for providing evaluation data tha serve to describe how students who use the new program do on some generally

known tests of achievement. This can be meaningful information to persons considering using the program. For example, a potential adopter may wish to have an answer to the question, "Can pupils show a respectable level of achievement on a standardized test if I adopt the IM program?" Perhaps of more importance, such test information can be used by the developer in identifying specific skills and abilities not being learned by pupils who use the program. This, in turn, should lead to a search for answers to such questions as: "Are these skills not taught by the program?" "If they are not taught, should the program be modified to teach them?" "If they are covered by the program but not being learned, what changes need to be made?" "Are these skills taught at a relatively late point in the new program and, therefore, not reached by pupils at this grade level?"

With the IM program, standardized test data have been used for such purposes as those described in the foregoing. For this reason, the scores can best be referred to as providing "descriptive evaluation data," as contrasted with "criterion evaluation data." A further qualification that must be noted by anyone examining these standardized test results is the fact that these data were obtained from schools that were also using other innovative programs. For example, as the IM program was being implemented and studied over a three-year period, changes were also being made in other aspects of the curricula of the two schools involved. These other changes, such as those in reading instruction, in the perceptual skills program, and in the science curriculum, could well have had an influence on math achievement.

Standardized test results that have been examined during the development of IM include results from two different tests used at School A, a small suburban school, and one test used at School B, a large inner-city elementary school.

School A Data

Scores from the Stanford Achievement Tests for School A primary grade pupils are presented in Table 1.

Table 1

Mean Scores for Arithmetic Subtests on SAT for School A Pupils, Grades 1-3, 1968-1973

Grade Level	Subtest	School Year				
		(Pre-IM)		(IM)		
		68-69	69-70	70-71	71-72	72-73
Grade 1	Arithmetic	2.2	2.4	2.4	2.5	2.5
Grade 2	Arithmetic Computation	2.8	2.7	2.9	3.0	2.9
	Arithmetic Concepts	3.2	3.0	3.4	3.7	3.2
Grade 3	Arithmetic Computation	3.4	3.7	4.1	3.8	3.9
	Arithmetic Concepts	4.3	4.6	4.7	4.8	4.7

The first two years for which data are reported are years in which IPI or a combination of PEP and IPI were used in these classrooms. IM was used in the years 70-71, 71-72, 72-73, although the first of these three years involved the implementation of a relatively incomplete version of IM. For the first grade, the data indicate that students were achieving above grade level prior to the introduction of IM and maintained this level after it was implemented. In the second grade, arithmetic computation skills remained at grade level with the new program while arithmetic concepts scores showed some increase. At the third-grade level, there was some improvement in computation scores for the IM years and a minor rise in concepts scores, which had been at a high level under the pre-IM program.

The scores on the Wide Range Achievement Test (WRAT) for School A pupils in kindergarten through third grade are summarized in Table 2.

Table 2

Mean Scores for the Arithmetic Subtest on the WRAT for School A Pupils, Grades K-3, 1969-1973

Grade Level	School Year			
	(Pre-IM)	(IM)		
	69-70	70-71	71-72	72-73
Kindergarten	1.5	1.9	1.6	1.4
Grade 1	2.5	2.5	2.6	2.5
Grade 2	2.9	3.2	3.4	3.2
Grade 3	3.9	4.2	4.1	4.1

Perhaps the most noticeable thing about the WRAT scores is the relatively high level of performance in kindergarten and first grade and a slightly lower relative performance for second and third grades, although pupils at the latter two grades are also scoring above grade level. The extent to which pupils show a particular grade-level performance on any standardized test is partly a function of the extent to which the abilities sampled by the test items are abilities covered at that grade level by these students. In an effort to examine possible causes of the relatively lower WRAT scores for pupils in grades 2 and 3 as compared with scores for pupils in grades K and 1, an analysis was made as to how far along in the IM curriculum a pupil would have to be to cover the content required to achieve given grade equivalents on the WRAT. Assuming that the average pupil worked through the IM units in the order in which they are numbered, the cumulative number of units that would have to be completed to achieve at each grade level can be summarized as follows:

Kindergarten	10 units
Grade 1	12 units
Grade 2	28 units
Grade 3	34 units

It can be assumed that the large number of additional units that had to be mastered to move from a grade equivalent near the beginning of second grade to that of completion of the second grade was a factor that produced the relatively smaller progression of second-grade pupils. That is, the WRAT data for School A could be interpreted as resulting from the fact that the average first grader enters that grade having mastered enough IM units to give him or her a grade equivalent of about 1.5 (approximately 11 units). Covering an additional 9 units in grade 1 (approximately 20 units in total) results in mastery of enough content to lead to a grade equivalent of about 2.5. However, mastering another 9 units in grade 2 (for a total of 29 units) would result only in a grade equivalent of about 3.1. In other words, covering the content required to go from 2.0 to 3.0 on the WRAT involves mastering a relatively large number of units (16) in the IM curriculum.

School B Data

The only standardized test scores used with the School B program were those provided by the WRAT. These are summarized in Table 3.

Table 3

Mean Scores for the Arithmetic Subtest on the WRAT for School B Pupils, Grades K-3, 1969-1973

Grade Level	School Year			
	(Pre-IM) 69-70		(IM)	
		70-71	71-72	72-73
Kindergarten	1.4	1.3	1.3	K.6
Grade 1	2.1	2.4	2.4	2.5
Grade 2	2.3	2.7	2.9	2.8
Grade 3	3.1	3.2	3.0	3.6

Here it can be seen that the kindergarten and first-grade scores are quite good, with the exception of the mean score of K.6 for the kindergarten in 72-73. This lower score is assumed to be associated with difficulties related to the implementation of certain non-math activities which resulted in some disruption of instructional schedules in these classrooms. The relatively smaller increase in performance from the end of first grade to the end of second grade (i.e., from 2.5 to 2.8 in 72-73) appears to be related to the larger number of units that must be mastered to cover the necessary content.

Discussion

The limited amount of data, gathered under previously qualified conditions, concerning the performance of IM pupils on standardized achievement tests may be summarized in terms of the following implications:

1. Pupils using the IM program can show relatively high achievement on standarized tests (note the data for grades K-3 at School A and for grades K and 1 at School B). The extent to which this result is attained is probably a function of exactly how the program is implemented.

2. The achievement of IM pupils on standardized tests is also a function of the match between the content of the test and the content of the IM units mastered. As indicated in the comparison of WRAT item content and IM units, second-grade pupils working in IM must master a relatively large number of units to move from a grade two to a grade three grade equivalent. Teachers using IM and concerned with scores on the WRAT as an important criterion measure could emphasize alternate paths through the hierarchy of units, paths that would insure that most of their second-grade pupils would master units covering skills measured by the test. This would mean that certain units, deemed

important by the IM development staff but not measured by the WRAT, might not be studied until third or fourth grade.

3. A curriculum which has the built-in flexibility of the IM program can be implemented by a local school staff in a manner designed to achieve the goals they consider most important. For example, the procedure suggested above for adapting sequences of units studied to enhance achievement on the WRAT could also be used with other tests as the criteria. On the other hand, a school might choose to ignore standardized test scores as criteria and use the program more as a vehicle for building pupil self-concept through experiences of meaningful progress and opportunities for independent planning, successful learning experiences on a daily basis, and objective self-evaluation.

THE PROCESS OF DEVELOPMENT AND IMPLEMENTATION: SOME CONCLUSIONS FROM WORK ON THE IM PROJECT

Work on the development and implementation of the Individualized Mathematics program involved the intensive efforts of a small project staff over a three-year period. Members of the staff, all of whom came to this work with a background of some involvement with earlier LRDC programs, had a relatively unique experience in terms of their personal participation in the broad range of tasks in design, development, implementation, and evaluation of the program. The scope of their work ranged from the specification of instructional objectives and the design of lessons and tests to extended experience in working with pupils and teachers both on instructional problems and on very practical problems of materials supply, storage, and replenishment. In view of the relative uniqueness of this experience, it would seem useful to attempt to summarize some of the insights concerning development, implementation, and evaluation, as these activities are carried out in ongoing school programs, that have been gained from this work.

Various procedures can be followed in developing and implementing a new instructional program. A common practice is for the developer to produce new materials, giving each item intensive tryouts with individual pupils, and then implement the program in one classroom in order to carry out any modifications that are suggested by this type of in-context trial. Following this latter trial, the revised program is turned over to persons interested in implementing it in ongoing school operations. The procedure used with the IM project can be considered as an extension of this common practice. This extension involved the participation of the development staff in the implementation of the program in 20 classrooms in 2 different schools, with the work extending over a 3-year period. The use of this approach was largely a recognition of the fact that implementing a program in a total school situation involves additional types of "development" that cannot be undertaken under the conditions of a more limited trial. Some of this development may involve modifications in instructional materials, the need for which only becomes evident when the program is used by a larger sample of teachers and students. However, much of the development has to do with problems of general management within a total school operation, problems of supply of materials, of teacher training, of classroom supervision, and of the integration of the new program with all other activities of the school day.

These were some of the kinds of problems encountered by members of the IM staff as they worked in the schools on a daily basis. They found that putting a program in general operation involved much more than providing teachers with materials and a set of guidelines or manuals for their use. What was required was additional work on program development.

The Tryout and Modification of Materials as an Aspect of In-School Development

Most of the lesson materials and tests and related record-keeping forms that were used in the implementation of the IM program had been given previous trials with pupils. Many of the materials were minor modifications of things used in earlier programs and had been given extended trial in actual classroom use. Those materials that were new and unique to IM were given a preliminary trial with individual students or small groups. However, it was recognized by the staff that information obtained in the initial large-scale implementation of all program components should be used as a basis for further improvement of materials and procedures. To accomplish this, a number of feedback mechanisms were used. These included in-class observations of students and teachers and records of pupil performance on lessons and tests. Results from this type of feedback have been described in the Evaluation section of this report. Our purpose here is to discuss some of the things learned about the processes of development and evaluation as a result of this experience.

Positive Outcomes

On the basis of the experience with IM, it would appear that a number of positive outcomes can be realized when developers are involved in the extended implementation of a new program. These include the following:

1. Developers have the opportunity to observe their program in a rather complete example of the total context in which it must ultimately function. It is being implemented by a variety of teachers with quite a broad sample of types of students.

2. Formal and informal feedback from teachers arises from each of these varied classroom contexts and is influenced by the differences in perceptions that each teacher brings to the task.

3. Teachers, because of the nature of their experiences and their specific priorities in the operation of an instructional program, attend to many aspects of the innovation that are probably not considered by the developers.

4. In the IM experience, teachers are very cooperative in providing feedback and offering suggestions. Having a relatively large number of persons quite intimately acquainted with details of tests and lesson materials provides a rich source for creative suggestions.

Some Problems

The experience that the IM staff had in extending their development activities into the initial implementation of the program served to suggest that this procedure also may result in certain problems. These include:

1. The presence of a program developer in the school and the classrooms, on a relatively regular basis, may reduce the extent to which the implementation can be considered typical of the eventual general implementation. For this reason, most LRDC programs are now given further trial and evaluation in "field site" schools where such intervention by developers is not a part of the operation.

2. With a developer on the scene, teachers may be inclined to bring to her or his attention problems which could more appropriately be solved by the teacher as an aspect of regular classroom operation. This is typically a result of the teacher's concern for doing a conscientious job in serving as a program evaluator.

3. The developer must learn to be not overly responsive to every teacher suggestion for change. Modifying materials or some other aspect of program operation in accordance with the suggestions of one teacher may serve to create new problems for other teachers. For this reason, suggestions made by one teacher must be checked with

other teachers before a change is implemented. If the change involves modification of lesson materials or other materials with which the pupil works, such changes should be tried with a sample of pupils before being incorporated into the total system.

4. The regular presence of a developer in the school may result in him being called upon to perform many services which should really be a part of the continuing responsibility of regular school staff.

5. Some of the steps that should be explored as a means to more effective operation of a new program probably cannot be implemented by a developer. Some such steps may necessitate coordinating activities used in teaching one subject with those involved in teaching other subjects. Others may require additional teacher time, time which can only be secured through action of a building principal or other responsible person. All of this seems to suggest the need for a further type of program "development" which can only be carried out by persons having a broader responsibility for the total school program than that which rests in the hands of the developers of any one program.

System Implementation and Modification as Aspects of In-School Development

The preceding section discussed aspects of basic program development--refining lesson materials, tests, and basic procedures for the use of such materials--as these development activities were facilitated by working within a large-scale school operation. This section will attempt to identify other aspects of program development, those related to overall operation of the system, to teacher training and supervision, to coordination with other school programs, and to the involvement of administrators, parents, and community.

IM as an Instructional System

The LRDC programs for the individualization of instruction in mathematics have used a type of "systems" approach to the development and the operation of the instructional program (Lindvall & Cox, 1970). That is, they are based on the assumption that effective individualization requires attention to all essential components of an instructional system. IPI, PEP, and IM have much in common in terms of the extent to which they have a "built-in" systematization of certain components. All three programs have prespecified objectives of instruction, a system of tests and of testing procedures, specific forms and procedures for prescription writing, and standardized teaching materials. In IPI, these materials and accompanying record forms, when used by teachers who had experienced a certain orientation to the program, are expected to result in the desired program operation in the classroom. PEP, in its most effective implementation, used most of the foregoing components but added a more intensive program of teacher training and continuing supervision in an effort to achieve more effective diagnosing and prescribing behaviors on the part of the teacher. IM, attempting to capitalize on the most effective aspects of IPI and PEP, endeavored to employ all of the major components of its predecessors. That is, it used most of the same type of "material" components of the earlier programs and attempted to add to and extend the attention to diagnostic and instructional procedures initiated by PEP. In this latter endeavor, however, certain modifications in procedure were necessary because of the fact that IM, rather quickly, was implemented in many more classrooms than had been the case with PEP. Among other things, this meant that the individual supervisory assistance provided under PEP had to be replaced by a reliance on documents offering guidelines and suggestions, such as the Teacher's Manual, the Testing Manual, and certain shorter documents outlining specific steps in classroom management. Also, IM represented an attempt to give attention to even more components

of the total instructional system than was the case with PEP and IPI. This expansion of the system concept and the attempt to implement it and study it in a relatively large number of classrooms provided the opportunity to gain new insights concerning this approach to program design and implementation. Some of these are described in the following section.

The Interaction Between the Roles of the Developer and Implementer

Perhaps the major lesson learned by the staff of the IM project from their work on the implementation of the program in ongoing school operations was that concerning practical limitations on the extent to which developers can impose a detailed instructional system upon a school. This situation may be analyzed in terms of the fact that even though developers concern themselves with the design of what would appear to be most of the relevant components of the instructional system, putting the program in operation in the school involves its interaction with a number of other "systems" already existing in the school and having a major influence on what takes place there. Here we are referring to such systems as the administrative structure of the school, traditional relationships between pupils and teachers, and such aspects of the social system as teacher role, general goals of teachers and administrators, and traditional norms for managing instruction. These considerations place certain limits on what the developer can accomplish as an implementer and suggest certain responsibilities that a program implementer may have for development.

The developer as an implementer. The IM experience seems to offer some support for the following perspectives on the task of the developer as an implementer:

1. An instructional program of any degree of complexity (such as IM, IPI, and PEP) will probably eventually be implemented in a

number of different ways, each representing a variation on the original specifications.

2. A program developer can only hope to develop and control some limited number of the components that will comprise the instructional system in operating schools and classrooms. One can only hope to certify that actual tryout shows that when the [illegible] components are used in this way and under these conditions, these results will be produced. The developer cannot hope to control all conditions of implementation nor hope to evaluate the program under all possible variations in the relevant conditions.

3. One given instructional program, depending upon how it is implemented, can be used to achieve different types of basic goals. For example, if a school using IM considered average pupil achievement on some specific standardized test as the major criterion of success, the local implementer could analyze the criterion test and then identify those components of the system that should be given major attention in adapting the program to this school (e.g., specific units to be emphasized, appropriate procedures for reinforcement for test passing, an emphasis on reinforcement for rapid progress, etc.). If another school had as a major goal the development of each pupil's self-confidence and self-initiative as a learner, other aspects of the program implementation would be emphasized.

4. A developer or implementer cannot expect to achieve the degree of implementation deemed essential without knowledge, and some degree of control, of a variety of components of the system that are already present within the school and community. As one simple example, a developer or implementer cannot expect a teacher to adopt a specific classroom procedure if this procedure is "frowned upon" by the administrative and supervisory staff of the school. The task of implementation, then, requires involvement and commitment on the part

of all persons having an impact on what takes place in classrooms. Some of this involvement may require certain types of development.

The implementer as a developer. The points outlined in the preceding section have specific implications for the task of implementation. Implementation is much more than the imposition of a given program operation upon a school and classrooms. It is the adaptation of the program and the modification of the local situation in such a manner as to maximize the chances that the given operation can achieve the desired local goals. In a real sense, the task of implementation is one of further on-site development. The following appear to be points that are relevant to this task:

1. There undoubtedly are some limits on the extent to which it is possible or desirable for a developer to impose all details of operation of an instructional system on a school or on an individual teacher.

2. The implementation role, whether carried out by an "implementation staff," by teachers, by administrators, or all of these persons, must involve further development activities. Of central concern here will be the development of teacher functions, of supervisory activities, and of a variety of procedures necessary for making the new program operate successfully in the local situation. Another major aspect of this development will be the specification of the goals of the local implementation and the modification of the program to focus on the achievement of these goals.

3. The LRDC School Implementation staff, working in two developmental schools, can only demonstrate implementation and development within these specific contexts. The staff can contribute to the development and improvement of the system and can demonstrate that programs do work under these certain conditions. Their role is analogous to that of the lesson designer giving new lessons a preliminary trial with one or two students at a time.

4. Another major contribution of the Center's School Implementation staff could well be in the development of guidelines for carrying out the implementation process in any local situation. This is another aspect of the implementer's role as a developer.

5. Implementation of a new program within any school situation probably should involve rather intensive self-study by the local staff. Such a study should seek to answer questions of the following type: (a) What goals are we attempting to achieve with this new program? (Studying this should involve a real effort to achieve staff consensus and commitment.) (b) Who are all the persons that must be involved if this new program is to be implemented in such a way as to achieve these goals? (c) In what ways must our school organization and procedures be modified if this program is to help us achieve our goals? (d) How must this innovative program be modified to serve our needs?

Summary

The development of the Individualized Mathematics program was carried out in an integrated development-implementation effort in which the program was being used in approximately 15 classrooms as it was being designed, tested, and revised. This integrated effort afforded an opportunity for the generation of certain tentative insights into the two processes involved. The basic insight would seem to be that development and implementation are not two discrete steps that can be carried out in a one-two order. Successful implementation requires certain types of adaptations of the new program to the requirements of the local school situation. It also requires certain adaptations of the school's total system of operation to the requirements of the new program. This means that development is not really completed until a program is operating successfully in each specific school context. It also implies that some key aspects of the development effort must be the responsibility of the implementer.

Reference Notes

1. Lindvall, C. M. Individualized mathematics teacher's manual. Unpublished manuscript, Learning Research and Development Center, 1971.

2. Lindvall, C. M. Individualized mathematics testing manual. Unpublished manuscript, Learning Research and Development Center, 1971.

3. Lindvall, C. M. Classroom management of the individualized mathematics program. Unpublished manuscript, Learning Research and Development Center, 1973.

4. Hosticka, A., Light, J., Lindvall, C. M., Meese, K., Miller, G., & Oles, A. The Learning Research and Development Center individualized mathematics project: The scope and goals of the current development program. Unpublished manuscript, Learning Research and Development Center, 1971.

5. Lindvall, C. M. Individualized instruction: Some suggestions for teachers and supervisors. Unpublished manuscript, Learning Research and Development Center, 1972.

References

Deskins, W. E. Abstract algebra. New York: Macmillan, 1964.

DiCostanzo, J. E. Integrating multiple scalogram analysis into a formative evaluation procedure for structuring a portion of an individualized mathematics curriculum. Unpublished master's thesis, University of Pittsburgh, 1974.

Dienes, Z. P., & Golding, E. W. Sets, numbers and powers. Essex: ESA, 1966.

Dienes, Z. P., & Golding, E. W. Approach to modern mathematics. New York: Herder & Herder, 1971.

Eves, H., & Newsom, C. V. An introduction to the foundations and fundamental concepts of mathematics. New York: Rinehart, 1958.

Flournoy, F. Elementary school mathematics. New York: Center for Applied Research in Education, Inc., 1964.

Glaser, R. Towards a behavioral science base for instructional design. In R. Glaser (Ed.), Teaching machines and programed learning II: Data and directions. Washington, D. C.: National Education Association, 1965.

Goals for school mathematics. The Report of the Cambridge Conference on School Mathematics. Boston: Houghton Mifflin, 1963.

Herstein, I. Topics in algebra. Boston: Blaisdell Publishing, 1964.

Hosticka, A. A design of a hierarchy of behavioral objectives for an elementary school mathematics curriculum. Unpublished doctoral dissertation, University of Pittsburgh, 1973.

Kelley, J. L. Number systems of arithmetic. The 69th Yearbook of the National Society for the Study of Education (Part 2), 1970, 69, 75-129.

Levi, H. Elements of algebra. New York: Chelsea Publishing, 1954.

Lindvall, C. M., & Bolvin, J. O. Programed instruction in the schools: An application of programing principles in "Individually Prescribed Instruction." The 66th Yearbook of the National Society for the Study of Education (Part 2), 1967, 66, 217-254.

Lindvall, C. M., & Cox, R. C. (with collaboration of Bolvin, J. O.). Evaluation as a tool in curriculum development: The IPI evaluation program. Chicago: Rand McNally, 1970.

Marks, J. L., Purdy, C. R., & Kinney, L. B. Teaching elementary school mathematics for understanding. New York: McGraw-Hill, 1970.

McFarland, D., & Lewis, E. M. Introduction to modern mathematics for elementary teachers. Boston: D. C. Heath & Co., 1966.

Morgan, M. M., & Paige, B. L. Algebra 1. New York: Holt, Rinehart and Winston, 1963.

Resnick, L. B., Wang, M. C., & Kaplan, J. Task analysis in curriculum design: A hierarchically sequenced introductory mathematics curriculum. Journal of Applied Behavior Analysis, 1973, 6, 679-710.

School Mathematics Study Group. Mathematics for elementary school, Book 1, Teachers commentary. Pasadena: A. C. Vroman, 1965.

Webb, E. J., Campbell, D. T., Schwartz, R. D., & Sechrest, L. Unobtrusive measures: Nonreactive research in the social sciences. Chicago: Rand McNally, 1966.

Wilder, R. L. Historical background of innovations in mathematics curricula. The 69th Yearbook of the National Society for the Study of Education (Part 2), 1970, 69, 7-22.

APPENDIX A

Specific Objectives for the Individualized Mathematics (IM) Program as Organized by Units

Unit 1. Matching: Same/More/Less

A. Given two sets of objects, the student pairs the objects and states if the sets have the same number of objects.

B. Given two sets of objects, one grossly larger than the other, the student states which set has more.

C. Given two sets of objects, one grossly larger than the other, the student states which set has less.

D. Given two sets of objects, the student pairs the objects and states which set has more.

E. Given two sets of objects, the student pairs the objects and states which set has less.

Unit 2. Seriation

A. Given two objects of different sizes, the student selects the larger (smaller) object.

B. Given three objects of different sizes, the student selects the largest (smallest) object.

C. Given objects graduated by sizes, the student seriates according to size.

D. Given three sets of objects, the student matches and states which set has the most (least).

E. Given several sets of objects, the student seriates the sets according to number of objects in each set.

Unit 3. Number Concept and Counting to 5

A. Given two sets of numerals (to 5), the student matches.

B. Given a stated numeral and printed numerals with sets to 5, the student selects the stated numeral.

C. Given a written numeral with a set to 5, the student reads.

D. The student recites numerals in order (to 5).

E. Given a set of 0-5 moveable objects, the student counts objects moving them out of set as he counts.

F. Given fixed sets of ordered objects 1 to 5, the student counts objects.

G. Given a stated numeral to 5, the student writes it.

Unit 4. Number Concepts and Counting to 10

A. Given two sets of numerals to 10, the student matches.

B. Given a stated numeral and printed numerals with sets to 10, the student selects the stated numeral.

C. Given a written numeral with a set to 10, the student reads.

D. The student recites numerals in order (to 10).

E. Given a set of 0-10 moveable objects, the student counts objects.

F. Given a fixed set of ordered objects 1-10, the student counts objects.

G. Given a stated numeral to 10, the student writes it.

Unit 5. Pairing Numerals with Sets to 5

A. Given a fixed set of unordered objects to 5, the student counts objects.

B. Given either a numeral to 5 and several sets of objects or a set and several numerals, the student either identifies the set represented by the numeral or the numeral represented by the set.

C. Given several sets of objects and several numerals to 5, the student matches numerals with the appropriate sets.

D. Given a number stated (to 5) and a set of objects, the student counts out subset as stated.

E. Given a number to 5 and a fixed set of objects, the student circles the number of objects represented by the number given.

Unit 6. Pairing Numerals with Sets to 10

A. Given a fixed set of unordered objects to 10, the student counts objects.

B. Given either a numeral to 10 and several sets of objects or a set and several numerals, the student either identifies the set represented by the numeral or the numeral represented by the set.

C. Given several sets of objects and several numerals to 10, the student matches the numerals with the appropriate sets.

D. Given a number stated (to 10) and a set of objects, the student counts out subset as stated.

E. Given a number to 10 and a fixed set of objects, the student circles the number of objects represented by the given number.

Unit 7. Comparison of Sets

A. Given two sets of objects, the student counts and states which set has more (less) objects or that the sets have the same number of objects.

B. Given three sets of objects, the student counts and states which set has the most (least).

C. Given two sets of objects (not paired), the student states which set has more--regardless of arrangement.

D. Given a set of objects and a numeral (to 10), the student states which shows more (less).

E. Given a set of objects and a set of numerals, the student selects the numeral that represents the set, one more than the set, and one less than the set.

F. Given two written numerals, the student states which shows more/less.

G. Given a set of numerals 0-10, the student places them in order.

Unit 8. Addition and Subtraction Concepts

A. Given two sets, the student makes the two sets equivalent by adding or removing objects.

B. Given two numbers (whose sum is to 9) and a set of objects with the directions to add, the student adds the numbers by counting out two subsets then combining and stating the combined number as the sum.

C. Given two numbers (to 9) and a set of objects with directions to subtract, the student counts out smaller subsets from larger sets and states remainder.

D. Given two sets, the student draws a picture showing the union of these sets. (LIMIT: Maximum of 9 elements in the union set.)

E. Given a set, the student shows (i.e., by crossing out objects) the removal of a subset.

F. Given a representation of union of sets or removal of a subset, the student writes the number sentence. (LIMIT: Sums through 5.)

G. Given a representation of union of sets or removal of a subset, the student writes the number sentence. (LIMIT: Sums from 6-9.)

Unit 9. Addition and Subtraction Using the Number Line

A. Given a number line illustrating a number sentence, the student identifies whether the operation of addition or subtraction is indicated. (LIMIT: Sums to 9.)

B. Given a number line and a completed addition or subtraction sentence, the student uses the number line to illustrate the sentence. (LIMIT: Sums through 9.)

C. Given two numbers stated (sums to 9) and a number line with directions to add, the student uses the number line to illustrate the addition and find the sum.

D. Given two numbers stated (to 9) and a number line with directions to subtract, the student uses the number line to illustrate the subtraction and find the solution.

E. Given a number line and an addition or subtraction problem, the student uses the number line to illustrate the operation and find the solution.

Unit 10. Addition and Subtraction Sentences

A. Given addition and subtraction word stories, the student writes the number sentence. (LIMIT: Sums to 9.)

B. Given pictures, the student illustrates union of sets or removal of a subset, and writes the number sentence. (LIMIT: Sums through 9.)

C. Given an addition or subtraction sentence in the form: $a + b = \square$ or $a - b = \square$, the student completes the sentence.

D. Given an addition or subtraction sentence in the form: $\overset{a}{\underline{+b}}$ or $\overset{a}{\underline{-b}}$, the student completes the sentence.

E. Given an addition sentence, the student writes two subtraction sentences for the same numbers. (LIMIT: Sums through 9.)

F. Given a number sentence in the form $a + \square = c$ or $\square + b = c$, the student completes the sentence. (LIMIT: Sums through 9.)

G. Given a number sentence in the form $a = \square + \triangle$, the student gives several solutions for the sentence. (LIMIT: Sums through 9.)

H. Given a set or a number line to represent a sum, the student writes several addition and subtraction sentences for each sum.

Unit 11. Beginning Fractions

A. Given a partitioned shape, the student indicates whether or not the parts are equivalent. (LIMIT: 2, 3, or 4 parts.)

B. Given a shape partitioned into equivalent parts, the student writes the numeral that indicates the number of parts. (LIMIT: 2, 3, or 4 parts.)

C. Given a whole object and fractional parts of that object, the student identifies one-half, one-third, one-fourth, and a whole object and states how many halves, thirds, or fourths are in a whole.

D. Given a set of objects partitioned into equivalent parts, the student states how many equivalent parts are in the whole and identifies whole, halves, thirds, or fourths of the set.

E. Given objects and sets of objects partitioned into equivalent parts, the student states how many equivalent parts are in the whole and identifies whole, halves, thirds, and quarters.

F. Given a shape, the student partitions it into halves, thirds, or fourths.

G. Given a set of objects, the student partitions it into halves, thirds, or fourths.

H. Given two wholes partitioned into halves, thirds, and quarters, the student states how many parts are in both wholes.

I. Given regions and sets, the student partitions them into halves, thirds, and quarters.

Unit 12. Money I

A. Given a collection of U. S. coins and the name of a specified coin, the student matches the specified coin with a picture of either of its faces.

Unit 13. Grouping by Tens and Ones

A. Given a set of objects, the student groups the objects by tens and counts the groups as one ten, two tens, three tens, . . ., ten tens. (LIMIT: Numbers of objects are in multiples of ten, from 10 through 100.)

B. Given a set of objects, the student groups the objects by tens and completes statements of the form _____ tens and _____ ones. (LIMIT: 1 through 99.)

C. Given objects structured in groups of tens and ones, the student completes statements of the form _____ tens and _____ ones, and then writes the standard numeral.

D. Given a list of standard numerals, the student selects the numeral named by a given statement of the form _____ tens and _____ ones. (LIMIT: 1 through 99.)

E. Given a standard numeral ≤ 100 and objects, the student counts out and arranges into sets of tens and extra ones the number of objects indicated by the numeral.

F. Given a standard numeral, the student completes statements of the form _____ tens and _____ ones. (LIMIT: 1 through 99.)

Unit 14. Time I

A. The student says the names of the days of the week in order.

Unit 15. Numeration to 100

A. Given a number chart, the student completes it by writing missing numerals from 1 through 99.

B. Given objects structured in groups of tens, the student counts the groups as ten, twenty, thirty, . . . , one hundred. The student says the multiples of ten in order from ten to one hundred.

C. Given a standard numeral, the student says the number name. (LIMIT: 1 through 100.)

D. The student counts to 100. (LIMIT: 1 through 100.)

Unit 16. Money II

A. Given a picture of a U. S. coin, the student writes its numerical value.

Unit 17. Addition and Subtraction with Sums to 18

A. Given an addition or subtraction sentence and a number line, the student uses it to illustrate and complete the sentence. (LIMIT: One-digit addends.)

B. Given an illustration of the union of sets or removal of a subset, the student writes the number sentence.

C. Given pictures, the student illustrates union of sets or removal of a subset and writes the number sentence. (LIMIT: One-digit addends.)

D. Given an addition or subtraction sentence, the student illustrates using union of sets or removal of a subset. (LIMIT: One-digit addends.)

E. Given an addition or subtraction word story, the student draws pictures illustrating union of sets or removal of a subset and writes the number sentence. (LIMIT: One-digit addends.)

F. Given an addition or subtraction sentence written in vertical form, the student completes it. (LIMIT: One-digit addends.)

Unit 18. Time II

A. The student says the names of the months in order.

B. Given a clock face, the student writes the missing numerals on the face.

C. Given a clock face with the minute and hour hands shown and a set of statements describing the position of each hand, the student identifies the correct statements for each face.

Unit 19. Open Sentences

A. Given an addition sentence, the student writes a second addition sentence which illustrates the commutative property of addition, the student tests to see if the commutative property holds for subtraction. (LIMIT: One-digit addends.)

B. Given an addition sentence, the student writes two subtraction sentences for the same numbers. (LIMIT: One-digit addends.)

C. Given a number sentence in the form $a = \square + \triangle$, the student completes the sentence in several ways. (LIMIT: One-digit addends.)

D. Given a number sentence in the form a + ☐ = c or ☐ + b = c, the student completes the sentence. (LIMIT: One-digit addends.)

E. Given a number sentence in the form a + b = ☐ + ☐, the student completes the sentence in several ways. (LIMIT: One-digit addends.)

F. Given a number sentence in the form a + b = c + ☐ or a + b = ☐ + d, the student completes the sentence. (LIMIT: One-digit addends.)

Unit 20. Multiplication

A. Given a set < 10, the student replicates and states "a sets of b are in c."

B. Given a number line with equal intervals indicated, the student completes statements of the form: There are _____ jumps of _____ in each jump.

C. Given a number line, the student begins at zero and draws arrows to show a given number of jumps of a given size.

D. Given an array, the student completes statements of the form: "There are _____ sets of _____ in each set."

E. Given the number of sets and the number in each set, the student makes the array. (LIMIT: One-digit factors.)

F. Given an array, or a number line with equal intervals indicated, the student completes statements of the form: "_____ jumps/rows/sets of _____ in _____." (LIMIT: One-digit factors.)

G. Given illustration of number line intervals and arrays, the student states that a sets of b = c can be noted as a x b = c. (LIMIT: Single-digit factors.)

Unit 21. Word Names--Zero to Ninety-Nine

A. Given word names for numbers zero to ten, the student reads the word name and matches it to the corresponding set or standard numeral.

B. Given a set or standard numeral 0 to 19, the student writes the word name for the number.

C. Given word names for multiples of ten < 100, the student reads the word name and matches it to the corresponding structured set or standard numeral.

D. Given word names for numbers to 99, the student reads the word name and matches to the corresponding structured set or standard numeral.

E. Given an ordered set with an element indicated, the student identifies the ordinal position of the element. (LIMIT: First-tenth.)

F. Given a pair of numbers, the student identifies which number is "greater than" and which number is "less than" the other or if they are equal and writes number sentences for the relationship.

Unit 22. Division

A. Given objects (20-100), the student groups by a number < 10 and states how many sets of a are in b.

B. Given a set of a objects, the student circles sets of b objects and completes statements of the form: "There are _____ sets of b in a.

C. Given a number line illustrating repeated subtraction, the student completes statements of the form: "_____ jumps of a in each jump were taken away from b to = zero" or "a jumps of _____ in each jump were taken from b to = zero." No repeated subtraction shown. (LIMIT: Products through 25.)

D. Given the number of objects in the array and the number of sets, the student makes the array and completes statements of the form "There are _____ sets of a in b" or "There are c sets of _____ in b." (LIMIT: One-digit factors.)

Unit 23. Applications

A. Given a pair of numbers and a specified operation, the student writes a number sentence or example and completes it. (LIMIT: Addition and subtraction of one-digit addends.)

B. Given an addition or subtraction word story, the student writes the number sentence for the specified numbers.

C. Given an addition or subtraction word problem involving money, measurement units, or time, the student writes the number.

D. Given a standard numeral or a Roman numeral, the student writes an equivalent Roman or standard numeral. (LIMIT: 1 through 30.)

Unit 24. Fractions

A. Given either a region partitioned into equivalent parts with shaded parts or the sentence, the student either completes sentences of the following form: _____ of _____ equivalent parts, or identifies the correct picture. (LIMIT: 12ths.)

B. Given either a region partitioned into equivalent parts with shaded parts or a fraction, the student either identifies the fraction which shows what part of the picture is shaded or identifies the correct picture. (LIMIT: 12ths.)

C. Given a region divided into equivalent parts, the student writes a fraction for the shaded part and identifies the numerator and the denominator. (LIMIT: 12ths.)

D. Given sets of objects or a region and a fraction, the student partitions them into equivalent parts and identifies the part the given fraction indicates. (LIMIT: 12ths.)

E. Given a pair of fractions, the student uses >, <, or = to show the relationship of the two fractions. (LIMIT: 12ths, and either the denominators or the numerators in a pair of fractions must be the same.)

Unit 25. Addition

A. Given three single-digit numbers, the student adds in two different ways to illustrate the associative principle for addition and tests for associative principle in subtraction.

B. Given column addition problems with 3 or 4 single-digit addends, the student solves and checks using the associative principle.

C. Given addition and subtraction problems with single-digit addends, the student demonstrates timed mastery.

Unit 26. Multiplication and Division

A. Given an array, the student writes multiplication sentences for each array. (LIMIT: Single-digit factors.)

B. Given an array, the student writes division sentences for the array. (LIMIT: Single-digit factors.)

C. Given word stories illustrating a multiplication sentence, the student writes the multiplication sentence. (LIMIT: Single-digit factors.)

D. Given word stories illustrating a division sentence, the student writes the division sentence. (LIMIT: Single-digit factors.)

E. Given a number line, the student shows counting by stated intervals to a given point and writes a multiplication sentence and a repeated addition sentence to match his illustration.

F. Given a number line and an end point, the student counts by stated intervals to the end point, answers the question, "How many intervals of A are in B?", and writes the problem in a division sentence and as repeated subtraction.

Unit 27. Numeration 100-1,000

A. Given structured groups of 100 objects in each group, the student counts by 100s to 900, and states that there are C groups of one hundred in C hundred.

B. Given a set of 100-1,000 objects (not round number but multiple of 10), the student counts by 100s, then 10s.

C. Given a set of 100-1,000 objects (not multiples of 10s or 100s), the student counts by 100s, then 10s, then ones.

D. Given structured groups to 999, the student completes the sentence _____ hundreds and _____ tens and _____ ones, and writes the number as a standard numeral.

E. Given a standard numeral for a number to 999, the student writes the digit which is in the ones', tens', or hundreds' place and identifies the place value of a particular digit.

F. Given a number $\leq$ 1,000, the student reads.

G. Given a stated number $\leq$ 1,000, the student writes.

H. Given word names for numbers to 999, the student reads the word name and matches it to the corresponding structured set or standard numeral.

I. Given a place value chart, the student writes numerals to 999 in columns for hundreds, tens, and ones according to the place value of each digit.

J. Given a stated number (100-1,000) and a set of objects, the student counts out correct subset and writes corresponding numeral.

K. Given an indicated seriation, the student writes numbers in short sequences of numerals from a starting point. (LIMIT: 999.)

Unit 28. Multiplication and Division Equations

A. Given a multiplication table for facts, both of whose factors $\leq$ 5, the student writes the numeral which belongs in each empty cell and uses the table to complete multiplication sentences of the form a x b = $\square$.

B. Given a multiplication table for facts with factors of 6-9, the student writes the numeral which belongs in each empty cell and uses the table to complete multiplication sentences of the form a x b = □.

C. Given multiplication sentences with a missing factor, the student solves the equation by writing the missing factor. (LIMIT: Single-digit factors.)

D. Given division sentences with a missing factor or product, the student fills in the frames for the missing term. (LIMIT: Single-digit factors.)

E. Given a division sentence, the student uses known multiplication facts to solve the division sentence. (LIMIT: Single-digit factors.)

F. Given a division sentence in the form □ ÷ b = c, the student writes a related multiplication sentence in the form b x c = □ or c x b = □ and completes both sentences by writing the missing product. (LIMIT: One-digit factors.)

G. Given a word problem for which a multiplication sentence is appropriate, the student writes a corresponding sentence and uses a multiplication table to complete it. (LIMIT: Two one-digit factors.)

H. Given a multiplication table and a word problem for which a division sentence is appropriate, the student writes the sentence using the multiplication table. (LIMIT: One-digit factors.)

Unit 29. Addition and Subtraction

A. Given a number written with two digits, the student expands and regroups.

B. Given an addition problem, the student adds using expanded notation. (LIMIT: ≤ 2-digit addends.)

C. Given a subtraction problem, the student subtracts using expanded notation. (LIMIT: Addend ≤ 2-digit numerals.)

D. Given an addition or subtraction problem, the student solves using expanded notation. (LIMIT: ≤ 2-digit numerals.)

Unit 30. Money III

A. Given a picture of a U. S. dollar bill, the student states how many coins of a specified denomination (penny, nickel, dime, quarter, or half-dollar) are equivalent in value to it. Given a picture of a five-dollar bill or a ten-dollar bill, the student states how many one-dollar bills are equivalent in value to it.

B. Given a picture of a U. S. coin, the student identifies a set of coins equivalent in value.

C. Given a picture of a dollar bill, the student identifies sets of coins that are equivalent in value to a dollar.

D. Given a picture of a set of coins, the student writes the value of the set and identifies a set of coins equivalent in value. (LIMIT: 99 cents.)

E. Given a picture of an item and its purchase price, the student identifies a set of coins needed to purchase it. (LIMIT: 99 cents.)

Unit 31. Addition and Subtraction--Two-Digit Numbers II

A. Given a two-digit addend and a one- or two-digit addend, the student adds using the standard algorithm. (LIMIT: No regrouping; sums to 99.)

B. Given a two-digit sum and a one- or two-digit addend, the student subtracts using the standard algorithm. (LIMIT: No regrouping; sums to 99.)

C. Given a two-digit addend and a one-digit addend, the student adds using the standard algorithm. (LIMIT: Sums to 99.)

D. Given a two-digit sum and a one-digit addend, the student subtracts using the standard algorithm.

Unit 32. Time III

A. Given a completed clock face with the minute hand at 12, the student completes statements of the form "_____ o'clock."

B. Given a completed clock face, the student completes statements of the form "_____ minutes past a o'clock." The student completes the statement, "There are _____ minutes in one hour." (LIMIT: 1 through 59 past the hour.)

C. Given a completed clock face, the student completes statements of the form "_____ minutes before _____ o'clock," and given a completed statement, he identifies the appropriate clock face. (LIMIT: 1 through 30 minutes before the hour.)

D. The student completes statements of the form, "There are _____ minutes in a (quarter, half) hour." Given statements of the form "a quarter past, a quarter to, or half past" a specified hour, the student selects the appropriate clock face.

E. Given a completed clock face, the student completes statements of the form "_____ minutes past _____ o'clock," and given a completed statement, he identifies the appropriate clock face. (LIMIT: 1 through 59 minutes past the hour.)

F. Given a completed clock face, the student completes two statements of the form "_____ minutes past _____ o'clock" and "_____ minutes before _____ o'clock." (LIMIT: 1 through 30 minutes before the hour.)

Unit 33. Addition and Subtraction--Two-Digit Numbers III

A. Given an addition or subtraction example, the student completes it using the standard algorithm and writes a corresponding example using the inverse operation. (LIMIT: Sums to 99; two one- or two-digit numerals.)

B. Given an addition or subtraction word problem, the student solves it. (LIMIT: Three one-digit addends or two two-digit addends; sums to 99.)

C. Given a two-digit addend and a one- or two-digit addend, the student adds using the standard algorithm and checks his answer using the commutative property. (LIMIT: Sums $\leq$ 198.)

D. Given a three-digit sum, the student subtracts using the standard algorithm to find the one- or two-digit difference. (LIMIT: Sums $\leq$ 198.)

E. Given an addition or subtraction problem whose addends are one- or two-digit, the student solves using the short algorithm. (LIMIT: Sums $\leq$ 198.)

Unit 34. Beginning Decimals

A. Given one of the following forms, a decimal fraction, word name, or common fraction for tenths, the student writes the other two forms.

B. Given any of the following forms, a decimal fraction, word name, or common fraction for hundredths, the student writes the other two forms.

C. Given a pure decimal fraction, the student writes it in expanded notation. (LIMIT: $\leq$ .99.)

Unit 35. Applications II

A. Given an addition or subtraction sentence illustrating the identity property of zero, or given a multiplication sentence illustrating the identity property of zero, the student completes the sentence by writing the missing addends, sum, factor, or product. (LIMIT: Sums to 99, or one-digit factors.)

B. Given a bar graph, the student answers questions and solves addition and subtraction problems based on the graph. (LIMIT: Sums to 99; one operation per problem.)

C. Given a standard numeral, the student identifies each expression equivalent to it in a list of indicated sums, indicated differences, indicated products, indicated quotients, or Roman numerals. (LIMIT: One operation per expression; sums to 99; one-digit factors; Roman numerals to XXX; multiplication table may be used for multiplication and division expressions.)

D. Given a multiplication or division word problem and a multiplication table, the student writes a completed number sentence appropriate to the solution of the problem. (LIMIT: One-digit factors; one operation per problem.)

E. Given a word problem and a set of number sentences, the student identifies a number sentence appropriate to the solution of the problem. (LIMIT: One operation per problem; sentence with the unknown isolated on the right side of the equation.)

F. Given two expressions each of which is an indicated sum, difference, product, or quotient, the student writes >, <, or = between them to complete the number sentence. (LIMIT: Sums to 99; one-digit factors; one operation per expression; multiplication and division expressions.)

G. Given a word problem, the student writes >, <, or = between them to complete the number sentence. (LIMIT: Sums to 99; one-digit factors; one operation per expression; a multiplication table may be used for multiplication and division problems.)

H. Given an addition, subtraction, multiplication, or division sentence, the student completes the sentence and then writes a corresponding number sentence for the inverse operation. (LIMIT: Sums to 99; one-digit factors multiplication table may be used for multiplication and division sentences.)

Unit 36. Numeration/Place Value

A. The student completes statements to indicate that ten ones equal one ten, ten tens equal one hundred, and ten hundreds equal one thousand.

B. Given a standard numeral, the student writes the digit that is in the ones', tens', hundreds', or thousands' place or identifies the place value of a specified digit.

C. Given a numeral, the student writes it in expanded notation; given the expanded notation for a numeral, the student writes the standard numeral.

D. Given a specified starting point, the student writes numerals in ascending order in short sequences by ones, fives, tens, or hundreds.

E. Given a numeral, the student writes it in expanded notation and renames it by regrouping from one specified place to another.

F. Given a number, the student rounds to the nearest ten or hundred.

Unit 37. Expanded Addition and Subtraction of Three-Digit Numbers

A. Given a three- or four-addend addition problem with a missing addend, the student solves for the missing term. (LIMIT: Single-digit addends.)

B. Given an addition or subtraction sentence with one- or two-digit addends and a missing sum or addend, the student completes it. (LIMIT: Sums to 99; two addends.)

C. Given any two-addend addition or subtraction problem where addends are $\leq$ three digits, the student solves using expanded notation and regrouping.

D. Given any two-addend addition problem whose addends are $\leq$ three digits, the student solves using partial sums.

Unit 38. Addition and Subtraction Using the Short Algorithm

A. Given a three-digit addend and a one-, two-, or three-digit addend, the student adds using the standard algorithm. (LIMIT: Three-digit sums.)

B. Given addends, the student adds using the standard algorithm. (LIMIT: Four three-digit addends.)

C. Given addends, the student adds using the standard algorithm. (LIMIT: Four four-digit addends; four digit sums.)

D. Given a three- or four-digit sum and an addend, the student subtracts using the standard algorithm. (LIMIT: Four-digit addends; no regrouping from hundreds' or thousands' place.)

E. Given a sum and an addend, the student subtracts using the standard algorithm. (LIMIT: Three- or four-digit addends; four-digit sums.)

F. Given an addition or subtraction example, the student solves using the standard algorithm and writes a corresponding example using the inverse operation. (LIMIT: Two four-digit addends; four-digit sums.)

Unit 39. Multiplication and Division Word Problems and Open Sentences

A. Given a division sentence of the form a $\div \square = c$, the student completes the sentence. (LIMIT: One-digit factors.)

B. Given a division sentence or example in which either the divisor or the quotient is 1, the student completes the sentence or example by writing the missing quotient, divisor, or dividend. (LIMIT: Dividend ≤ 999.)

C. Given a word problem for which division is the appropriate operation, the student writes the corresponding division fact and writes the answer with the appropriate label. (LIMIT: One-digit factors.)

D. Given a word problem for which multiplication is the appropriate operation, the student writes a corresponding multiplication fact and writes the answer with the appropriate label. (LIMIT: Two one-digit factors.)

E. Given a number, the student writes a specified number of basic multiplication facts with the given number as the product. (LIMIT: Given number must be nonzero product of two one-digit factors.)

Unit 40. Mastery of Multiplication and Division Facts

A. The student demonstrates mastery of the basic facts of multiplication. (LIMIT: One-digit factors; horizontal and vertical form.)

B. The student demonstrates mastery of the basic facts of division. (LIMIT: One-digit factors; forms b/a, a ÷ b, and a/b.)

C. Given an array partitioned to show the partial products resulting from the multiplication of a two-digit number by a one-digit number and given directions for writing a multiplication sentence for each part of the array, the sum of the partial products, and a multiplication sentence for each part of the array, the sum of the partial products, and a multiplication sentence for the entire array, the student writes each of the sentences and the sum. (LIMIT: Two-digit factors less than 20.)

D. Given multiplication problems, the student writes the multiplicand in expanded notation and solves it. (LIMIT: 9 x 99.)

E. Given a multiplication example with a two-digit multiplicand and a one-digit multiplier, the student multiplies using partial products.

F. Given a division example with a two-digit dividend and a one-digit divisor, the student solves using partial quotients. (LIMIT: No remainder.)

G. Given three one-digit factors, the student multiplies.

H. Given a multiplication example with a three-digit multiplicand and a one-digit multiplier, the student multiplies using partial products.

I. Given a division example with a three-digit dividend and a one-digit divisor, the student solves using partial quotients. (LIMIT: No remainder.)

J. Given multiplication example with a two-digit multiplicand and a one-digit multiplier, the student multiplies using the multiplication algorithm.

K. Given a multiplication example with a three-digit multiplicand and a one-digit multiplier, the student multiplies using the multiplication algorithm.

Unit 41. Fractions

A. Given a picture of a region or a set of objects partitioned into equivalent parts, the student writes the common fraction for a specified fractional part.

B. Given a common fraction a/b and several pictures of a region partitioned into b congruent parts, the student shades each region to illustrate the fraction so that no two regions look the same.

C. Given a common fraction and a picture of a set of objects, the student circles a specified fractional part of the set.

D. The student completes statements or common fractions to tell how many halves, thirds, fourths, . . . or tenths are equivalent to two wholes.

E. Given a common fraction greater than or equal to 2, the student writes the equivalent whole number or mixed form with the fraction less than 1.

Unit 42. Fractions: Addition and Subtraction

A. Given two fractions < 1 with the same denominator, the student adds and states if the sum is >, <, or = to 1.

B. Given two fractions with the same denominator, the student subtracts. (LIMIT: Addend ≤ 1.)

Unit 43. Money IV

A. Given a picture of a set of U. S. coins and bills or a statement of the value of some set (e.g., two dollars and thirty-five cents), the student writes the value of the set using the dollar sign and decimal point. (LIMIT: $10.00; 10 coins in a pictured set.)

B. Given a picture of a set of coins and bills and an item with a specified purchase price, the student indicates whether there is not enough, just enough, or more than enough money in the set to purchase the given item. (LIMIT: $10.00; 10 coins in a pictured set.)

C. Given the price of a purchase, the amount of money given to the salesman, and a picture of a set of coins, the student identifies those coins which could be received as change. (LIMIT: $10.00 purchase price; 10 coins in a pictured set.)

D. Given an addition or subtraction problem involving money values, the student solves it and labels his answer with the cent sign or with the dollar sign and decimal point. (LIMIT: One operation per problem; 4 addends; sums to $10.00.)

E. Given a picture of two sets of coins and bills, the student writes >, <, or = between the sets to show the relationship of the value of the two sets. (LIMIT: $10.00; 10 coins in a pictured set.)

Unit 44. Time IV

A. Given a clock face without hands and a time stated either as "a minutes before b" or "C minutes past d," the student draws the hour hand and the minute hand to show the given time. (LIMIT: 1 through 30 minutes before the hour.)

B. Given a time statement of the form "a minutes past b," the student writes the time using the form "b:y." Given a time statement of the form "b:y," the student writes an equivalent time statement by completing a statement of the form "_____ minutes past _____."

C. The student completes statements of the form "There are _____ hours in one day." Given a completed clock face labeled to indicate the part of the day, the student writes the time in the form x:y with the appropriate designation--a.m., p.m., noon, or midnight.

Unit 45. Applications

A. Given a number sentence illustrating the commutative or associative property of addition or multiplication, the student identifies the property. Given the name "commutative" or "associative" and a list of operations, the student identifies those operations for which the specified property holds. (LIMIT: Any given number sentence illustrates a single use of a single property; operations--addition, subtraction, multiplication, division.)

B. Given an addition, subtraction, multiplication, or division sentence in which a missing addend, sum, factor, or product is represented by n, the student solves for the value of n. (LIMIT: Sums to 99; a one-digit factor and a one- to two-digit factor.)

C. Given an expression containing more than one operation, with the order of operations indicated by parentheses, the student writes the standard numeral for the expression. (LIMIT: Four numerals per expression; no numeral occurs within more than one set of parentheses; operations--addition, subtraction, multiplication, division; sums to 99; a one-digit and a one- to two-digit factor.)

D. Given a set of number sentences and a word problem which requires two different operations, the student identifies a number sentence appropriate to the solution of the problem. (LIMIT: Operations--addition, subtraction, multiplication, division; an operation may be used only once; sentences with the unknown isolated on the right side of the equation.)

E. Given an addition, subtraction, or multiplication word problem and a list of possible answers, the student rounds the given addends, sum and addend, or the larger factor to estimate his answer and then selects the correct estimated answer from the given list. (LIMIT: Four four-digit addends; sums of four digits; one-digit factor times a three-digit factor; rounding to the nearest ten or one hundred; no rounding to the ten thousands' place.)

F. Given a picture or a description of finite set and a capital letter to name the set, the student defines the set by writing a statement of the form A = {a, b, c, d, e, f}. (LIMIT: 0 to 10 elements.)

G. Given a finite set defined by a statement of the form A = {a, b, c, d, e, f} and a list of possible subsets, the student identifies those sets which are subsets of the given set. (LIMIT: Possible subsets must be nonempty.)

H. Given data and a grid with the scale indicated, the student draws and labels bars on the grid to complete a bar graph for the given data. (LIMIT: One set of data per graph; 5 bars per graph.)

I. Given a line graph or pictograph, the student answers questions and solves addition and subtraction word problems based on the graph. (LIMIT: Sums to 9, 999; one operation per word problem.)

J. Given a Roman numeral, the student writes the equivalent standard numeral. (LIMIT: 2,000.)